**The Dissertation Committee for Daniel Lee Murphy Certifies that this is the approved version of the following dissertation:**

# The Effects of Serial Correlation on the Curve-of-factors Growth Model

**Committee:**

S. Natasha Beretvas, Co-Supervisor

Keenan A. Pituch, Co-Supervisor

Brandon K. Vaughn

Tiffany A. Whittaker

Daniel A. Powers

# The Effects of Serial Correlation on the Curve-of-factors Growth Model

by

**Daniel Lee Murphy, B.A.; M.Ed.**

**Dissertation**

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

**Doctor of Philosophy**

**The University of Texas at Austin**

**May, 2009**

## Dedication

Dedicated to my wife

Sharon Frankson,

and our children

Annie and Jack

## Acknowledgements

There are several individuals I would like to thank for their guidance and support throughout my graduate school experience. First, I would like to thank three professors who provided me with intellectual and emotional support, and enabled my graduate school years to be productive and enjoyable: Keenan Pituch, Tasha Beretvas, and Barbara Dodd. These three professors have been outstanding role models for the quantitative methodologist I aspire to become.

Keenan, as my dissertation co-chair, has also always been available to discuss my dissertation, as well as any other research project on which I happened to be working. He is the person who introduced me to growth modeling, which has influenced the direction of my research and my career. I am also indebted to Keenan for his assistance with my first publication. The research projects on which I have been fortunate enough to collaborate with him have been invaluable as learning and growth experiences.

Tasha, as my other dissertation co-chair, has also always been available to discuss my dissertation or any other academic issue of interest. I have been fortunate to have attended several classes with Tasha; she has been an inspirational teacher who has had a positive influence on the direction of my research and the quality of my work. Learning about factor analysis from Tasha provided me with the foundational knowledge that enabled me to begin the research that resulted in this dissertation.

Barbara has been a constant source of support throughout my time in graduate school; she has helped me with academic and non-academic issues too numerous to list. I have learned a tremendous amount about quantitative methods from Barbara through attending her classes and through research collaborations. I will be forever grateful for all her efforts to facilitate my success.

I would also like to thank the other members of my committee, Brandon Vaughn, Tiffany Whittaker, and Dan Powers. Their assistance has enabled what can be a difficult process to go very smoothly. Brandon Vaughn I would like to thank for his availability and willingness to discuss various research topics of interest. I am also grateful for the projects on which we have collaborated. I would like to thank Tiffany Whittaker for introducing me to structural equation modeling, which eventually led to the topic of my dissertation. Dan Powers I would like to thank for his in depth instruction on the topic of growth models.

I would like to thank my family. I am indebted to my wife's parents, Courtland and Marie Frankson. Their generosity and availability made what could have been a disruptive time to my immediate family much less stressful. My sister Carol and her family, my sister Patti, and my brother Michael have also been generous and instrumental in helping my wife, my children, and myself to cope successfully during this time of change.

I am extremely grateful to my mother and father, Edna and Robert Murphy, for fostering an enthusiasm for learning in me. Thanks, mom and dad, for providing me with the intellectual and emotional resources necessary to fulfill my goals. I would also like to thank you for your encouragement of my education in general and of my decision to return to school in particular.

Finally, I would like to thank my wife, Sharon, and our children, Jack and Annie. From the day that I decided to return to graduate school they have provided me with unwavering support, despite the demands that took my time away from them. They never complained of the sacrifices they needed to make in order to help me fulfill my goals. Thank you Sharon, Annie, and Jack; my accomplishments would not have been possible without your patience, understanding, and generosity.

# The Effects of Serial Correlation on the Curve-of-factors Growth Model

Publication No._____________

Daniel Lee Murphy, Ph.D.

The University of Texas at Austin, 2009

Supervisors: S. Natasha Beretvas and Keenan A. Pituch

This simulation study examined the performance of the curve-of-factors growth model when serial correlation and growth processes were present in the first-level factor structure. As previous research has shown (Ferron, Dailey, & Yi, 2002; Kwok, West, & Green, 2007; Murphy & Pituch, 2009) estimates of the fixed effects and their standard errors were unbiased when serial correlation was present in the data but unmodeled. However, variance components were estimated poorly across the examined serial correlation conditions. Two new models were also examined: one curve-of-factors model was fitted with a first-order autoregressive serial correlation parameter, and a second curve-of-factors model was fitted with first-order autoregressive and moving average serial correlation parameters. The models were developed in an effort to measure growth and serial correlation processes within the same data set. Both models fitted with serial correlation parameters were able to accurately reproduce the serial correlation parameter and approximate the true growth trajectory. However, estimates of the variance components and the standard errors of the fixed effects were problematic. The two

models also produced inadmissible solutions across all conditions. Of the three models, the curve-of-factors model had the best overall performance.

# Table of Contents

# List of Tables

## List of Figures

# Chapter I: Introduction

Longitudinal studies serve many purposes in education and psychology. Examples of interest to educational and psychological researchers include developmental attainment, change in risk factors over time, growth-rates of educational skill levels, and onset and cessation of behaviors (Browning, Leventhal, & Brooks-Gunn, 2004; Chatterji, 2006; Garner & Raudenbush, 1991; Sampson, Raudenbush, & Earls, 1997). The existence of many longitudinal panel data sets (i.e., data that tracks the same set of subjects across repeated observations) tracking children, adults, communities, organizations, etc., has increased interest in statistical models that measure change across time.

There are a variety of statistical techniques available with which to model change across time. Some commonly used models include repeated measures multivariate analysis of variance (MANOVA), autoregressive or time-series models, multilevel models, and latent growth models. These models differ in how well they capture the research questions of interest and how well the assumptions underlying the models match the empirical data (Curran & Bollen, 2001). The decision as to which model will best fit the data is critical, and it is rarely clear cut.

Two models from the structural equation modeling (SEM) framework that have received considerable attention in the social sciences are the autoregressive time series model and the latent growth curve (sometimes called latent trajectory) model (Bollen & Curran, 2004). Traditionally researchers have attempted to identify the conditions under which the growth curve and time series approaches do or do not fit empirical longitudinal data (Bast & Reitsma, 1997; Curran, 2000; Kenny & Campbell, 1989; Marsh, 1993; and

Rogosa & Willett, 1985). This comparative approach has inadvertently fostered an either/or perspective with regard to modeling longitudinal data. If the data are assumed to be a series of correlated events measured across time, then autoregressive models are considered to be more appropriate. In contrast, if the data can be assumed to be a series of independent events measured across time, then latent growth models are considered to be more appropriate.

A third type of structural equation model that has attracted recent interest is the curve-of-factors model (Leite, 2007), which is sometimes called a second-order latent growth model. In contrast with the latent growth model which models growth as a first-order factor, the curve-of-factors model models growth as a second-order factor. The first-order factors in a curve-of-factors model are indicated by multiple manifest variables that are measured repeatedly across time, and the second-order factors indicate the initial factor level of interest and the shape of the growth curve. Two major advantages of the curve-of-factors model are: 1) the factors can be considered to be "true" scores because the measurement error is modeled, and 2) measurement invariance across subgroups can be evaluated. In contrast, the growth curve models mentioned previously assume that the measurements are invariant across subgroups and no measurement error exists (Leite).

As mentioned above, the difference between the time series approach and the latent growth approach relates to the issue of serial correlation. An autoregressive time series model specifies recently measured variables as a function of earlier measurements. The variables are considered to be correlated across time, with variables closer together in time more highly correlated than those further apart. For example, in a first- order autoregressive model, the first and second observations would be more highly correlated than the first and third observations. This correlation structure is assumed to be the same for all individuals.

In contrast, latent growth models allow each participant to have an individual growth trajectory. Latent growth models usually assume that the series of measurements are independent, and serial correlation is considered to be a nuisance variable. Failure to model serial correlation when it is present in the data has been shown to bias latent growth curve and multilevel growth curve model parameters of interest (Ferron, Dailey, & Yi, 2002; Kwok, West, & Green, 2007; Murphy & Pituch, 2009; Sivo, Fan, & Witta, 2005). The effects of unmodeled serial correlation within the curve-of-factors model have yet to be studied.

This dissertation revolved around two related goals. First, the effects of unmodeled or mismodeled serial correlation within the curve-of-factors model were examined. Second, the curve-of-factors model was combined with two different autoregressive processes, a first-order autoregressive process [AR(1)] and a first-order autoregressive-moving average process [ARMA(1, 1)] in an attempt to integrate the models.

The goal of the first three chapters is to orient the reader to the structure and common assumptions of SEM in general, and to the latent growth model, the multiple indicator ARMA (1, 1) time series model, the autoregressive latent trajectory model, and the curve-of-factors model in particular. The curve-of-factors model is illustrated, and then it is integrated with the time series models. The remainder of the dissertation will describe the research design, which utilized Monte Carlo methods to investigate the impact of unmodeled and mismodeled serial correlation within the curve-of-factors model framework.

# Chapter II: Literature Review

This study investigates problems that can occur when researchers make invalid assumptions about the relationships among variables that are measured repeatedly. Specifically, the problems associated with ignoring or mismodeling serial correlation in longitudinal panel data is examined. This section of the dissertation will orient the reader to SEM in general and the models examined in this study in particular.

*Structural Equation Models*

The structural equation approach simultaneously estimates relations between observed variables and their corresponding underlying constructs, and between the latent constructs themselves (Bentler, 1980; Curran & Bollen, 2001). The terms latent factors and latent constructs are used interchangeably in this dissertation to refer to the unobserved and unobservable traits that are indicated by responses to observed variables. The structural equations used to estimate relations between observed variables and latent constructs can be thought of as consisting of three separate but interrelated models: a factor analytic or measurement model, a structural model, and a means model.

*Measurement Model*

The measurement or factor analytic model relates the observed variables to the underlying latent constructs such that,

$$\mathbf{y} = \boldsymbol{\tau} + \boldsymbol{\Lambda}_{\mathbf{y}}\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbf{y}$ is a vector of observed variables, $\boldsymbol{\tau}$ is a vector of measurement intercepts, $\boldsymbol{\Lambda}_{\mathbf{y}}$ is a matrix of factor loadings relating the observed variables to the matrix of latent constructs

$\boldsymbol{\eta}$, and $\boldsymbol{\varepsilon}$ is a vector of measurement residuals. Thus, in the measurement model the observed variable values are considered to be a function of unobservable latent constructs and measurement error. The distribution of latent constructs and the relationships between latent constructs are described in the structural model.

*Structural Model*

The structural model can be described as,

$$\boldsymbol{\eta} = \boldsymbol{\mu} + \boldsymbol{\zeta} \tag{2}$$

where $\boldsymbol{\mu}$ is a vector of latent construct means and $\boldsymbol{\zeta}$ is a vector of disturbances. The disturbances represent individual deviation about the latent construct means. This is the structural model that will apply to the growth models described later in the dissertation. For example, in a latent growth model the vector of latent construct means would consist of the mean of the level factor ($\mu_\alpha$) and the mean of the shape factor ($\mu_\beta$). The vector of disturbances would represent individual variation about the mean of the level and the mean of the shape. The disturbances are represented by the covariance structure $V(\zeta) = \boldsymbol{\Psi}$.

The structural model can also describe the latent constructs as a function of other latent constructs (Bentler, 1988) as in the following equation,

$$\boldsymbol{\eta} = \boldsymbol{\beta\eta} + \boldsymbol{\zeta}, \tag{3}$$

where $\boldsymbol{\beta}$ is a matrix of structural regression coefficients that relate latent factors to one another, and $\boldsymbol{\zeta}$ is a vector of disturbances as described above. This is the structural model that will apply to the quasi-simplex models described later in the dissertation. Under the quasi-simplex framework, each latent construct is regressed on the previous latent construct. The $\boldsymbol{\beta}$ matrix would contain the regression coefficients relating each construct to its preceding construct. As with Equation 2, the disturbances are represented by the covariance structure $V(\zeta) = \boldsymbol{\Psi}$.

*Means Model*

Whereas the measurement model estimates relationships between observed variables and latent constructs and the structural model estimates relationships between the constructs themselves, the means model estimates parameters associated with the constructs. Given that $\boldsymbol{\mu_y}$ is the vector of observed means and $\boldsymbol{\mu}$ is the vector of latent factor means, the means model structure is given as,

$$\boldsymbol{\mu_y} = \boldsymbol{\Lambda_y \mu} \tag{4}$$

where $\boldsymbol{\Lambda_y}$ is a matrix of factor loadings. For structural models that regress current factors on previously measured factors (e.g., the quasi-simplex), the means model can be expressed as,

$$\boldsymbol{\mu_y} = \boldsymbol{\nu} + \boldsymbol{\Lambda}(\boldsymbol{I} - \boldsymbol{\beta})^{-1}\boldsymbol{\eta}, \tag{5}$$

where $\boldsymbol{\nu}$ is a vector of intercepts, $\boldsymbol{I}$ is the identity matrix, and the other terms are as defined above.

*Covariance Structure*

When estimating the relations between the observed variables and the latent constructs, structural equation models recreate as closely as possible the relationships (i.e., the variances and covariances) among the original observed variables. The variance and covariance structures formed by the residuals and disturbances described in Equations 1 – 3 combine to form the model-implied covariance structure,

$$\boldsymbol{\Sigma}_{yy} = \boldsymbol{\Lambda}(I - \boldsymbol{\beta})^{-1}\boldsymbol{\Psi}(I - \boldsymbol{\beta})^{-1^{T}}\boldsymbol{\Lambda}^{T} + \boldsymbol{\Theta}_{\epsilon}, \tag{6}$$

where $\boldsymbol{\Lambda}$, $\boldsymbol{\beta}$, and $\boldsymbol{\Psi}$ are as defined above, and $\boldsymbol{\Theta}_{\epsilon}$ is the covariance matrix of the measurement residuals (i.e., the ε in Equation 1).

The models described in this dissertation will all be presented in terms of SEM. Thus, although the particulars of the models will differ, the basic three-part framework of a measurement model, a structural model, and a means model described above will apply to all models. Before structural equation models can be estimated, identification must be established. This topic is described in the following section.

*Identification*

Identification refers to whether or not enough information is supplied by the observed variables to provide unique values for the parameters that are to be estimated within the model structure. The information supplied by the observed variables includes the means, variances, and covariances of the random variables (i.e., the observed measurements). The information supplied by these random variables is referred to as the known parameters (Bollen, 1989). If the unknown parameters in a particular portion of a model (e.g., measurement model, structural model, means model) are unique functions of the known parameters, their identification is established. In cases where there is a unique solution for each model parameter and where there are more observed means, variances, and covariances than there are model parameters, the model is said to be overidentified. If the number of unknowns and knowns are equal, the model is said to be just identified. If the number of unknowns outnumbers the number of knowns, the modeled is said to be underidentified, meaning identification cannot be established. If all parameters in a model are uniquely identified, meaning the model is just identified or overidentified, identification is established and model estimation is the next step.

As an example, for the linear growth model which is presented in detail later in the paper, the unknown parameters are the intercept mean ($\mu_\alpha$), the slope mean ($\mu_\beta$), the intercept variance ($\psi_{\alpha\alpha}$), the slope variance ($\psi_{\beta\beta}$), the covariance between the intercept and the slope ($\psi_{\alpha\beta}$), the error variance ($\Theta_\varepsilon$), and the factor loadings ($\lambda_t$). As mentioned above, one necessary condition for model identification is that there are at least as many parameters known to be identified as there are unknown parameters to be estimated.

In general there is a population mean ($\mu_{yt}$) for each measurement occasion ($y_{it}$) as well as the population variances and covariances of the $y_{it}$s. This results in $T(T+3)/2$ identified parameters with which to work, where $T$ is the number of observations recorded across time. Without further restrictions, there will not be enough information provided to identify all of the growth model parameters. One common assumption that helps to alleviate this problem is the assumption that the trend values specified by the factor loadings of the slope factors ($\lambda_t$) are known (Bollen, 2004). An example of this would be the linear trajectory where the intercept is modeled as the initial set of measurements. In this case, the initial slope factor loading $\lambda_1 = 0$ and each consecutive $t$ measurement occasion would increase incrementally as specified by the researcher. For instance, a linear trajectory for four observations where the intercept is modeled as the initial set of measurements would result in $\lambda_t = 0, 1, 2, 3$. Thus, $\lambda_t$ slope factor loadings would be known and not estimated.

A second common assumption is that the error variances are independent, meaning they are uncorrelated but can take on different values across time (Sivo et al., 2005). Under these two assumptions, there are now $T + 5$ unknowns for the linear model (i.e., the $T$ error variances, $u_\alpha$, $u_\beta$, $\psi_{\alpha\alpha}$, $\psi_{\beta\beta}$, and $\psi_{\alpha\beta}$). The necessary condition for identification will not be satisfied with only two waves of data because there are $T + 5 = 7$ unknown parameters and only five known parameters with which to work (i.e., two observed means, two observed variances, and one observed covariance). In cases such as this when there is insufficient information available to provide a unique solution for each model parameter, the model is said to be underidentified.

Three waves of data do satisfy the necessary condition of identification. There are nine known parameters (three observed means, three observed variances, three observed covariances). The parameters of interest that require identification in a growth curve model with three waves of data are the mean intercept and mean slope, the variances and covariance of the intercepts and slopes, and three error variances, leading to a total of eight unknown parameters. Since there are more known than unknown parameters, the model is considered to be overidentified.

The fact that a model is considered identified as a whole is a necessary but not sufficient condition of identification. Each separate portion of the model (i.e., the mean model, the structural model, the measurement model) must be identified in addition to the overall model. In other words, there must be more observed variables than estimated parameters for each portion of the structural equation model considered separately as well as for the model considered as a whole in order for identification to be established.

### *SEM Assumptions*

In order to estimate SEM parameters accurately, some basic assumptions must be met. Notably, the outcome variable is assumed to be measured on a continuous scale and is also assumed to be multivariate normally distributed (Bentler, 1988). However, these two assumptions can be relaxed if methods other than maximum likelihood estimation are used (Bentler). The models in this dissertation are also restricted to structural models where the variables are assumed to be linearly related. While it is possible to model categorical outcomes and nonlinear relations between variables, such models were not considered in this dissertation. The next section of the dissertation describes the growth models that were examined.

### *Growth Models*

#### *Unconditional Latent Growth Model*

Latent growth models attempt to answer the broad question: What is the mean trajectory of the sample across time? The trajectory can be considered to be composed of two latent factors: an initial average level, and an average shape – which can be linear or curvilinear – representing the rate of change across time. The mean level and mean shape of the trajectory of the sample are often referred to as the fixed effects of the model.

However, individual subjects in the sample may have trajectories that are increasing or decreasing at a different rate than the sample as a whole. The degree to which it is necessary to model the trajectories of individual subjects is a question of inter-subject differences in intra-subject change.

Inter-subject change refers to the degree to which individual trajectories deviate from the trajectory parameter estimates, i.e., the degree to which the individual values deviate about the mean values of the level and shape factors. The inter-subject deviations from the trajectory parameter estimates are summarized by the covariance structure of the level and shape (e.g., the variance of the level, the variance of the shape, and the covariance between the two factors). The point estimates of this covariance structure are often referred to as the random-effects components of the trajectory model (Bollen, 2004).

The unconditional linear latent growth model models the fixed-effects components with the following notation (Bollen, 2004):

$$y_{it} = \alpha_i + \lambda_t \beta_i + \varepsilon_{it} \tag{7}$$

where $y_{it}$ is the value of the observed variable $y$ for the $i^{th}$ subject at time $t$; $\alpha_i$ can be thought of as the random intercept for subject $i$; and $\beta_i$ can be thought of in the linear model as the random slope for subject $i$. The $\lambda_t$ represent the repeated measures time points. There are a variety of ways in which to model time via $\lambda_t$. The $\lambda_t$ is commonly modeled as a constant, and a common coding convention is to have $\lambda_1 = 0$, $\lambda_2 = 1$ with subsequent $\lambda_t$ increasing in increments of one. By setting $\lambda_1 = 0$, $E(\alpha_i)$ represents the mean of the trajectory at the initial time point. Note that nonlinear change can be modeled by adding additional $\lambda_t \beta_i$ parameters (Bollen, 2004).

The error variable ($\varepsilon_{it}$) represents variability unrelated to the latent level and shape factors. It is a combination of random measurement error and systematic unique factors. The error scores can be allowed to correlate across time, although typically they are modeled as independent (i.e., uncorrelated) with mean zero (Sivo et al., 2005) as shown in the covariance matrix for the error scores presented below,

$$\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_t^2 \end{bmatrix}. \tag{8}$$

If the error scores were allowed to correlate across time, the zero terms in the off diagonals of the matrix would be estimated. One method of modeling correlation among the errors involves the specification of autocorrelation processes or functions, which will be discussed later in the dissertation. Previous research has shown that mismodeling the error covariance matrix in growth models can result in biased estimates of the fixed effects and the variance of the random effects (Ferron et al., 2002; Kwok et al., 2007; Murphy & Pituch, 2009; Sivo et al., 2005).

*Fixed effects and random effects*. The fixed effects estimates of the mean intercept and mean slope are of interest and are modeled with the following notation (Bollen & Curran, 2004):

$$\alpha_i = \mu_\alpha + \zeta_{\alpha i} \tag{9}$$

$$\beta_i = \mu_\beta + \zeta_{\beta i} \tag{10}$$

where $\mu_\alpha$ and $\mu_\beta$ are the mean intercept and mean slope across all subjects. The intercept equation represents the intercept of subject $i$ ($\alpha_i$) as a function of the mean of the intercepts for all subjects ($\mu_\alpha$) and a disturbance $\zeta_{\alpha i}$. Similarly, the slope equation treats the slope of subject $i$ ($\beta_i$) as a function of the mean of the slopes for all subjects ($\mu_\beta$) and a disturbance $\zeta_{\beta i}$. Equation 9 presents the variance covariance matrix of the intercept and slope disturbances,

$$\mathbf{\Psi} = \begin{bmatrix} \sigma^2_{\zeta_{\alpha_i}} & \sigma_{\zeta_{\alpha_i}\zeta_{\beta_i}} \\ \sigma_{\zeta_{\alpha_i}\zeta_{\beta_i}} & \sigma^2_{\zeta_{\beta_i}} \end{bmatrix} \tag{11}$$

where $\sigma^2{}_{\zeta\alpha i}$ is the variance of the mean intercept factor, $\sigma^2{}_{\zeta\beta i}$ is the variance of the slope factor, and $\sigma_{\zeta_{\alpha_i}\zeta_{\beta_i}}$ is the covariance between the intercept and slope factors.

The intercept and slope equations can be combined into a single equation by substituting the right-hand sides of Equations 9 and 10 for $\alpha_i$ and $\beta_i$ respectively into Equation 7. The resulting equation is

$$y_{it} = \left(u_\alpha + \lambda_t u_\beta\right) + \left(\zeta_{\alpha i} + \lambda_t \zeta_{\beta i} + \varepsilon_{it}\right). \tag{12}$$

Equation 12 illustrates that the trajectory of $y_{it}$ is a function of the intercept mean, the growth trend variable ($\lambda_t$) multiplied by the slope mean, and a composite disturbance term. Note that the presence of $\lambda_t\zeta_{\beta i}$ makes the composite disturbance heteroscedastic over time, because the variance depends on $\lambda_t$ which changes as a function of time. The first term in parentheses is often referred to as the fixed component and the second term the random component (Bollen & Curran, 2004).

In matrix form, the fixed component can be described in terms of the means model presented in Equation 4 where $\boldsymbol{\mu_y}$ refers to the observed variable means, $\boldsymbol{\Lambda}_y$ is a $j$ x 2 (for the linear model) matrix of factor loadings, and $\boldsymbol{\mu}$ is a 2 x 1 vector containing the means $\mu_\alpha$ and $\mu_\beta$ respectively.

The terms in the random component combine to create the model-implied covariance matrix as presented in Equation 6, where $\boldsymbol{\Psi}$ is the 2 x 2 covariance matrix of the intercept and slope factors presented in Equation 11, and $\boldsymbol{\Theta}_\varepsilon$ is the covariance matrix of errors presented in Equation 8.

*Measurement model*. A sample of individual trajectory equations as presented in Equation 12 can also be summarized in matrix form as the measurement model expressed in Equation 1 (Singer & Willett, 2003), where $\mathbf{y}$ is a vector of observed scores for subject $i$ across measurement times one to $j$, $\boldsymbol{\Lambda}_\mathbf{y}$ is a $j$ x 2 (for the linear model) matrix containing a column of intercept coefficients and a column of slope coefficients, $\boldsymbol{\eta}$ is a vector containing the latent intercept ($\alpha_i$) and slope ($\beta_i$) scores, and $\boldsymbol{\varepsilon}$ is a vector of error scores for each of the measurement occasions. The expanded form of Equation 1 for the measurement portion of the latent growth model is thus,

$$\begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{ij} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \\ \vdots & \vdots \\ 1 & \lambda_j \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} + \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{ij} \end{bmatrix}. \tag{13}$$

*Structural model*. The intercept and slope means presented in Equations 9 and 10 can be summarized in terms of the structural model illustrated in Equation 2, where for the linear model $\boldsymbol{\mu}$ is a vector containing the means of the intercept and slope factors and $\boldsymbol{\zeta}$ is a vector of the disturbances. The expanded matrix form of the structural part of the linear latent growth model is thus,

$$\begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} = \begin{bmatrix} \mu_{\alpha_i} \\ \mu_{\beta_i} \end{bmatrix} + \begin{bmatrix} \zeta_{\alpha_i} \\ \zeta_{\beta_i} \end{bmatrix}. \tag{14}$$

An unconditional latent growth model with five measurement times is presented in Figure 1.

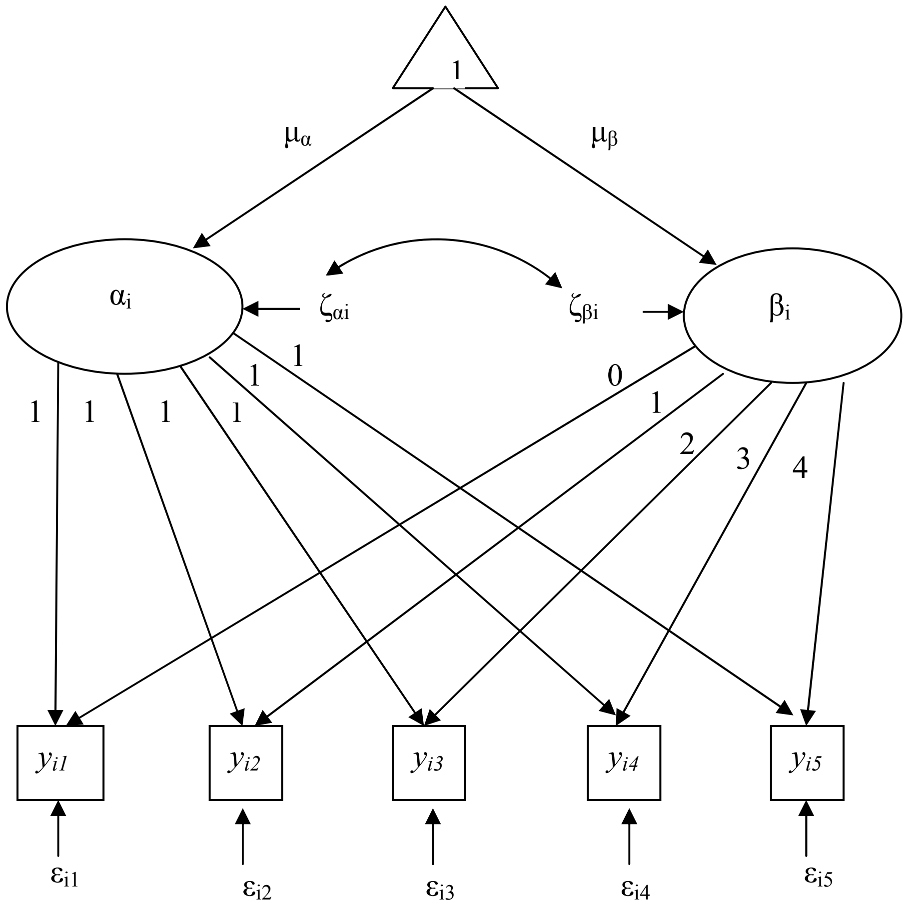


Figure 1. Unconditional latent growth model.

*Assumptions and limitations*. The latent growth models described in this dissertation commonly assume an equal number of assessments for all individuals, the assessments are equally spaced, and that there is no missing data. Some of the earlier resources on latent growth models (Willet & Sayer, 1994; Stoolmiller, 1995) indicate that these assumptions must be met in order to estimate latent growth models. However, more recent developments in model estimation, such as full-information maximum likelihood (FIML) estimation, enable these assumptions to be eased (MacCallum, Kim, Malarkey, & Kiecolt-Glaser, 1997).

Maximum likelihood estimation is a commonly used estimation procedure whereby a likelihood function is maximized in terms of the moments of the data, meaning data is input in the form of a covariance or correlation matrix. The more recent FIML estimation procedure defines the likelihood function in terms of individual scores on observed variables. The use of individual scores as input data enables the model to be fit to whatever data are available for each subject. FIML estimation is now available in most commercial software programs for fitting structural equation models (Curran & Bollen, 2004).

A critical assumption of the latent growth model is that all cases have the same functional growth form (Hertzog & Nesselroade, 2003), which is specified by the shape factor. Individual cases are free to vary in the rate of change within a specific functional form, but the form itself cannot vary across cases. For example, it is not possible to specify a latent growth model where some of the cases exhibit linear growth while others exhibit non-linear growth.

The observed variables used to measure growth are often in the form of multiple-item composite variables, meaning observed variable scores are in the form of item sums or means (Leite, 2007). When measuring growth using multiple-item composites, the first-order latent growth model makes an assumption of strict factorial invariance. Strict factorial invariance is a term that defines the intercepts, factor loadings, and error variances of all items as being equivalent at different times of measurement (Herzog & Nesselroade, 2003; Leite; Meredith, 1993). It is a very restrictive assumption that may be difficult to meet in practice.

Leite (2007) showed that the first-order latent growth model produces unbiased estimates of the mean of the shape, and the variances and covariance of the level and shape if the items are essentially tau-equivalent. Essentially tau equivalent items have equal factor loadings but different error variances and intercepts. However, if the items had different factor loadings, error variances, and intercepts (i.e., were essentially congeneric) the latent growth model was found to produce negatively biased estimates of the mean of the shape, as well as of the variances and covariance of the level and shape. Further, estimates of the mean of the level were positively biased under both conditions.

A final assumption of the latent growth model commonly made in practice is that the manifest variable errors are uncorrelated across time, as specified in Equation 8. An advantage of latent growth models is the ease with which this assumption is relaxed. In addition to modeling the covariance between pairs of manifest errors directly, it is also possible to model serial correlation processes that are commonly found in time series models.

According to Sivo et al. (2005), when growth curve models are fitted to longitudinal data, researchers should consider modeling autoregressive (AR), moving average (MA), and autoregressive moving average (ARMA) processes in the manifest error structure. Unmodeled serial correlation may diminish the ability of a researcher to detect growth as well as bias parameter estimates of interest. Although there have been some recent attempts to combine time series and latent growth models (Bollen & Curran, 2004; Curran & Bollen, 2001, Sivo et al., 2005), traditionally time series models have been considered as an alternative to growth curve modeling. The next section will consider such an alternative, specifically the quasi-simplex model (Rovine & Molenaar, 2005; Sivo, 2001; Sivo & Willson, 2000).

*Quasi-Simplex Models*

According to Rovine and Molenaar (2005), the use of simplex models to account for the correlation structure of a series of measurements dates back to the early defining article by Guttman (1954). The prominent characteristic of this correlation structure is that the correlations decrease as they move away from the diagonal of the structure. Moving away from the diagonal of the correlation structure also corresponds to increasing temporal distance in longitudinal panel data. Jöreskog (1970) then described these models in terms of structural equations. He distinguished between the perfect simplex structure which exists only if the measurement errors of the items measured are negligible, and the quasi-simplex structure which includes a measurement model. In this section of the dissertation I will consider the multiple indicator ARMA(1, 1) model. Note that by constraining the MA(1) parameter to zero, the ARMA(1, 1) model reduces to the AR(1) model.

Traditionally, researchers have compared the fit of growth curve models and time series models to the same set of data as a means of identifying the underlying trends of the data. Both models will specify a trend across time. As opposed to the latent growth models which assume two constructs (i.e., level and shape) are responsible for individual differences across time, the quasi-simplex assumes that a single construct is responsible for growth. The structure of time series models in general and the quasi-simplex model in particular permit explanatory variables to be incorporated at specific measurement occasions. This feature makes these models well suited to modeling different factors affecting the interindividual variance at different times (Bast & Reitsma, 1997). In contrast, the latent growth model growth is represented by a constant base of initial factor levels which change as a function of time. Explanatory variables are incorporated in order to explain individual differences in the initial factor level and rate of change, rather than to explain differences at specific measurement occasions.

In contrast to the individual factor differences estimated using latent growth models, time series models assume stability of factors across time. A subject's initial factor level will determine his or her position throughout the series of measurements. Differences in growth are a function of prior factor level and random disturbances. Because of the different perspectives on the nature of interindividual change, researchers have traditionally viewed the fitting of latent growth models and time series models to longitudinal data as an either/or proposition.

The quasi-simplex is a specific type of time series model that can model two stochastic processes common to time series models in general – an autoregressive (AR) process and a moving average (MA) process. In the AR(1) quasi-simplex, each observation is modeled as a function of its correlation with the preceding observation. In an MA(1) quasi-simplex, each measurement error is modeled as a function of its correlation with the previous measurement error. An ARMA(1, 1) quasi-simplex combines these two serial correlation processes. Although single indicator quasi-simplex models exist, this section of the dissertation will describe multiple indicator quasi simplex models.

*Measurement model*. The measurement model of the quasi-simplex can be expressed as in Equation 1, where the vector $\mathbf{y}$ contains $t$ sets of variable indicators measured across time, $\boldsymbol{\tau}$ is a vector of measurement intercepts, $\boldsymbol{\Lambda}$ is a matrix of loadings relating each of the $\eta_t$ constructs to its measured variable indicators, $\boldsymbol{\eta}$ is a vector of the $\eta_t$ constructs, and $\boldsymbol{\varepsilon}$ is a vector of random normal errors. The expanded form of Equation 1 for the multiple indicator quasi-simplex model for a single subject, where five latent factors (for five time points) are indicated by four manifest variables, is presented below in Equation 15. Note that the factor loading of the first manifest variable at each measurement occasion is set to 1 for scaling purposes.

$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \\ y_{41} \\ y_{12} \\ y_{22} \\ y_{32} \\ y_{42} \\ y_{13} \\ y_{23} \\ y_{33} \\ y_{43} \\ y_{14} \\ y_{24} \\ y_{34} \\ y_{44} \\ y_{15} \\ y_{25} \\ y_{35} \\ y_{45} \end{bmatrix} = \begin{bmatrix} 0 \\ \tau_{21} \\ \tau_{31} \\ \tau_{41} \\ 0 \\ \tau_{22} \\ \tau_{32} \\ \tau_{42} \\ 0 \\ \tau_{23} \\ \tau_{33} \\ \tau_{43} \\ 0 \\ \tau_{24} \\ \tau_{34} \\ \tau_{44} \\ 0 \\ \tau_{25} \\ \tau_{35} \\ \tau_{45} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \lambda_{21} & 0 & 0 & 0 & 0 \\ \lambda_{31} & 0 & 0 & 0 & 0 \\ \lambda_{41} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda_{22} & 0 & 0 & 0 \\ 0 & \lambda_{32} & 0 & 0 & 0 \\ 0 & \lambda_{42} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_{23} & 0 & 0 \\ 0 & 0 & \lambda_{33} & 0 & 0 \\ 0 & 0 & \lambda_{43} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_{24} & 0 \\ 0 & 0 & 0 & \lambda_{34} & 0 \\ 0 & 0 & 0 & \lambda_{44} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda_{25} \\ 0 & 0 & 0 & 0 & \lambda_{35} \\ 0 & 0 & 0 & 0 & \lambda_{45} \end{bmatrix} \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \eta_{i3} \\ \eta_{i4} \\ \eta_{i5} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{41} \\ \varepsilon_{12} \\ \varepsilon_{22} \\ \varepsilon_{32} \\ \varepsilon_{42} \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{33} \\ \varepsilon_{43} \\ \varepsilon_{14} \\ \varepsilon_{24} \\ \varepsilon_{34} \\ \varepsilon_{44} \\ \varepsilon_{15} \\ \varepsilon_{25} \\ \varepsilon_{35} \\ \varepsilon_{45} \end{bmatrix} \quad (15)$$

*Structural model*. The structural model of the ARMA(1, 1) quasi-simplex model can be expressed as,

$$\boldsymbol{\eta}_{\mathbf{t}} = \boldsymbol{\beta}_{\mathbf{21}}\boldsymbol{\eta}_{\mathbf{t-1}} + \boldsymbol{\xi}_{\mathbf{t}} + \boldsymbol{\Psi}\boldsymbol{\xi}_{\mathbf{t-1}}, \quad (16)$$

for $t$ = 1 to $T$ occasions. In this equation $\boldsymbol{\eta}_{\mathbf{t}}$ represents the vector of latent variables, $\boldsymbol{\beta}_{\mathbf{21}}$ represents a matrix of regression coefficients between adjacent latent factors measured on occasions $t - 1$ and $t$, $\boldsymbol{\xi}_{\mathbf{t}}$ represents a vector of disturbances modeled as latent error factors for each occasion, and $\boldsymbol{\Psi}$ represents a matrix of regression coefficients between adjacent latent errors for occasions $t - 1$ and $t$. For five measurement occasions, the multiple indicator ARMA(1, 1) model can be expressed by the following expanded matrix equation (Sivo, 2001):

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \beta_{21} & 0 & 0 & 0 & 0 \\ 0 & \beta_{21} & 0 & 0 & 0 \\ 0 & 0 & \beta_{21} & 0 & 0 \\ 0 & 0 & 0 & \beta_{21} & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{bmatrix} + \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 \\ \gamma_{67} & 1.0 & 0 & 0 & 0 \\ 0 & \gamma_{67} & 1.0 & 0 & 0 \\ 0 & 0 & \gamma_{67} & 1.0 & 0 \\ 0 & 0 & 0 & \gamma_{67} & 1.0 \end{bmatrix} \begin{bmatrix} \eta_6 \\ \eta_7 \\ \eta_8 \\ \eta_9 \\ \eta_{10} \end{bmatrix} \qquad (17)$$

The AR(1) portion of the ARMA(1, 1) process is represented by the $\beta_{21}$ regression coefficients in the $\boldsymbol{\beta}_{\mathbf{21}}$ matrix. These coefficients represent the association between each factor and the previously measured factor. The MA(1) portion of the ARMA(1, 1) process is represented by the $\gamma_{67}$ in the $\boldsymbol{\Psi}$ matrix. These coefficients represent the association between each latent error and the latent error of the previously measured factor. Note that the regression coefficients of each autocorrelation process are constant across time and across subjects.

A multiple indicator ARMA(1, 1) quasi-simplex model for five measurement occasions where each measurement occasion is indicated by four manifest variables is presented in Figure 2.

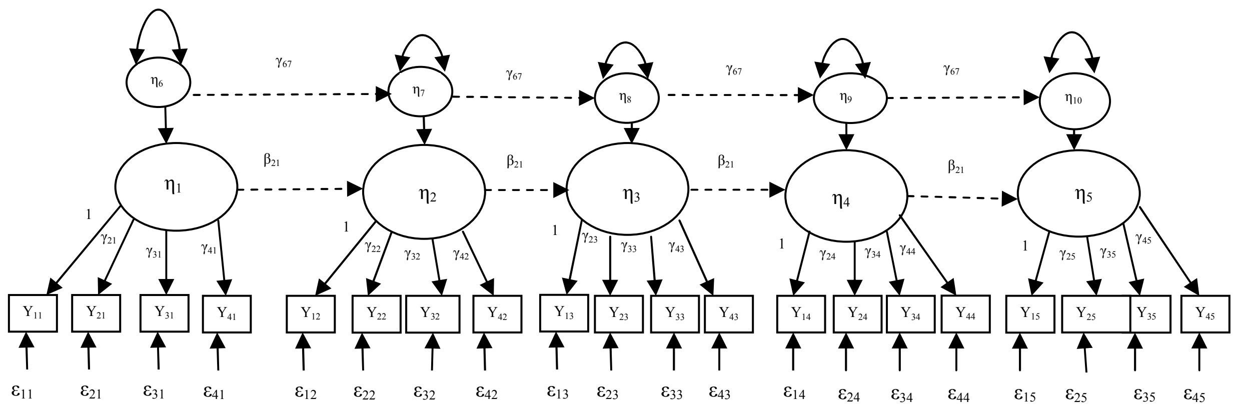


Figure 2. ARMA(1, 1) quasi-simplex model.

*Assumptions and limitations.* The quasi-simplex model described in this dissertation assumes equal spacing of measurement occasions, equal reliabilities, equal stabilities, and equal variances of the latent construct across time (Rudinger & Reitz, 2001). Reliability under the SEM framework refers to the relationship between the latent factors and the observed variables. Specifically, it refers to the quality of measurement of the construct of interest. This quality of measurement is assumed to be equal across time. In contrast, stability refers to the relationships between constructs at the structural level. Specifically, stability refers to the correlation between the latent factors. The quasi-simplex is assumed to have what Tisak and Meredith (1990) refer to as monotonic stability, meaning individuals maintain their same rank order across the measurement occasions.

An example of monotonic stability can be found in the Matthew effect hypothesis. The Matthew effect hypothesis refers to "the rich getting richer and the poor getting poorer." For example, the Matthew effect in reading predicts that the gap between good and poor readers increases with time (Bast & Reitsma, 1997). According to the hypothesis, individual variation in reading across time will be characterized by stable rank ordering of individuals and an increase in performance differences across time. The quasi-simplex model is well suited to modeling the Matthew effect.

Note, however, that the latent growth models can also model the Matthew effect if the level is modeled as the initial set of measurements and the correlation between the level and shape is high. Thus, although monotonic stability is not an assumption of latent growth models, latent growth models can model monotonic stability. This led Bast and Reitsma (1997) to compare the fit of latent growth models and quasi-simplex models in their examination of the Matthew effect with respect to the reading growth of elementary school students.

Growth curve models would be expected to fit the longitudinal data better if the rates of growth were variable, and the initial level of reading achievement did not necessarily determine the trajectory of the growth. In contrast, autoregressive models would be expected to fit the data better if the students maintained their position across time and the latent variability at each measurement occasion was relatively low.

Comparative approaches similar to that conducted by Bast and Reitsma (1997) have tended to foster a polarization of views that have led many proponents of one modeling approach to reject the methods of the other and vice versa (Bollen & Curran, 2004). However, what has become increasingly apparent is that there is not necessarily a "right" or "wrong" approach to analyzing repeated measures data and that it is possible for both types of growth processes to be found in the same longitudinal panel data set

(Bollen & Curran). Given that the autoregressive and growth models are each associated with key advantages and disadvantages, there has been some recent interest into combining the two approaches (Curran & Bollen, 2001; Sivo et al., 2005). Further, unmodeled serial correlation may cause growth curve model parameters to be poorly estimated (Sivo et al.). The following section presents the autoregressive latent trajectory (ALT) model which allows both growth curves and autocorrelation to be modeled simultaneously (Bollen & Curran; Curran & Bollen).

*Autoregressive Latent Trajectory Model*

The autoregressive latent trajectory (ALT) model includes the random intercept and random slope factors of the latent growth model in order to capture the fixed and random effects of the underlying growth trajectories. The ALT model also includes the standard fixed autoregressive parameters in order to capture the time-specific influences among the repeated measures themselves. It is possible for both processes to be present within the same longitudinal data set (Bollen & Curran, 2004; Curran & Bollen, 2001).

For example, Hussong, Hicks, Levy, and Curran (2001) studied the relation between variations in daily alcohol use and daily mood fluctuations over a 30-day period. In their study, the standard AR model does not allow for the hypothesized individual specific random components (i.e., the random intercepts and slopes) underlying each of these processes. However, the standard latent growth model does not incorporate the hypothesized time-specific lagged effects between alcohol use and mood across each day of measure. Thus, neither modeling strategy by itself allows for a comprehensive test of the hypothesized model.

*Measurement model*. The ALT measurement model equation for the set of repeated measures on construct $y$ as presented by Bollen & Curran (2004) is

$$y_{it} = \alpha_i + \Lambda_t \beta_i + \rho_{t,t-1} y_{i,t-1} + \varepsilon_{it}, \tag{18}$$

where $t$ = 2, 3, …, $T$, $\alpha_i$ and $\beta_i$ are the random intercepts and slopes as defined in the latent growth model, and $\rho_{t,t-1}$ is the correlation between adjacent measurements as defined in the quasi-simplex model. This model also assumes that the $\varepsilon_{it}$ are non-autocorrelated but note that this restriction can be removed. As this equation illustrates, the ALT model permits lagged values of $y$ to influence current values at the same time that the trajectory of $y$ is in part influenced by the random intercepts and slopes. Thus, the key features of the autoregressive and latent trajectory models are present in a single equation.

*Structural model*. As with the structural model of the standard latent trajectory curve described in the latent growth curve section, the random intercept and slope components can be expressed in terms of Equation 14, where the fixed $\begin{pmatrix} \mu_{\alpha_i} \\ \mu_{\beta_i} \end{pmatrix}$ and random $\begin{pmatrix} \zeta_{\alpha_i} \\ \zeta_{\beta_i} \end{pmatrix}$ trajectory components now represent the trend after the lagged autoregressive effects have been taken into consideration.

Although the ALT model may appear to be a simple combination of the latent growth model and the AR quasi-simplex model, there is a mitigating circumstance. Complications emerge because it is conceivable that the autoregressive function extends prior to the first wave of data, meaning the first wave of data would be dependent on a previous wave of data. A simple way to avoid the complications is to treat $y_{i1}$ as predetermined. If the $y_{i1}$ is treated as predetermined in the ALT model, the $y_{i1}$ can be expressed simply by an unconditional mean and an individual deviation from the mean (Bollen & Curran, 2004, Curran & Bollen, 2001). Specifically,

$$y_{i1} = v_1 + \varepsilon_{i1}\,, \tag{19}$$

where $v_1$ is the unconditional mean of the first set of measurements. The predetermined $y_{i1}$ can correlate with $\alpha_i$ and $\beta_i$.

However, there are some instances where treating the initial measure as endogenous rather than predetermined will be required to achieve identification. When the initial measure is not considered predetermined, then non-linear factor loadings for $y_{i1}$ must be estimated. The ALT model with an endogenous initial measure is beyond the scope of this dissertation and will not be considered further. The interested reader is referred to Bollen and Curran (2000, 2004).

The ALT with a predetermined initial measure is presented in Figure 3.

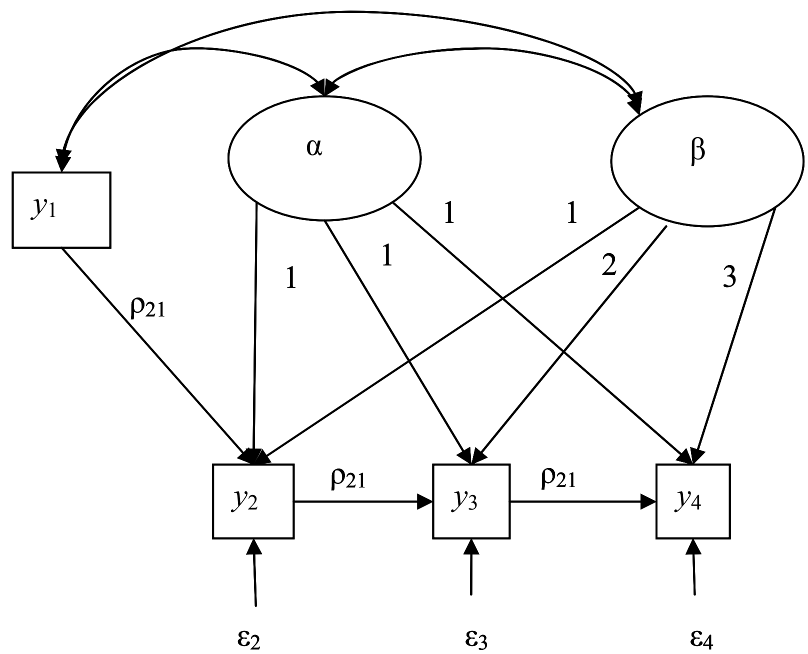


Figure 3. Autoregressive latent trait (ALT) model with predetermined first measure.

*Unconditional Curve-of-factors Model*

The latent growth model and the ALT model both measure a single multiple-item variable across time. It is conceivable that this single indicator could be broken down into smaller composite variables. For example, scores from the Texas Assessment of Knowledge and Skills (TAKS) tests are reported as a single measure; however, each test contains questions that measure sub-domains of interest. The 5$^{th}$-grade science consists of 13 questions measuring the nature of science, 9 questions measuring life science, 9 questions measuring physical science, and 9 questions measuring earth science. In this case the science instrument could conceivably be measured as separate composite variables by considering each sub-domain to be a separate measurement instrument.

As described above, when modeling growth using a composite variable such as the TAKS science test, the latent growth model and the ALT model must assume strict factorial invariance with regard to the sub-domains. If this assumption is not met, it can lead to biased estimates of parameters of interest (Leite, 2007). In contrast, each sub-domain of the test can be modeled as a separate indicator of a science knowledge factor under the curve-of-factors model.

The curve-of-factors model is sometimes referred to as a second-order latent growth model because the growth is modeled as a second-order factor. The latent growth model presented earlier is a first-order latent growth model. The first-order factors of the curve-of-factors model are latent constructs that are measured by multiple indicator variables. The constructs are measured across multiple time points. This common factor part of the model consists of latent constructs, manifest indicators, and measurement error. In theory, by accounting for measurement error the curve-of-factors model provides an error-free construct for growth modeling (Hancock, Kuo, & Lawrence, 2001).

*Measurement model*. As with the other structural equation models presented in this dissertation, the measurement portion of the model can be expressed in terms of Equation 1, where $\mathbf{y}$ is a vector of indicator variables, $\boldsymbol{\tau}$ is a vector of indicator intercepts, the matrix $\boldsymbol{\Lambda}$ specifies the factor loadings relating the indicator variables to the first-order latent constructs, $\boldsymbol{\eta}$ is a vector of the latent constructs, and $\boldsymbol{\varepsilon}$ is a vector of random normal errors (i.e., measurement error). An example of the expanded form of Equation 1 would be the same as the expanded matrix form of the measurement model of the quasi-simplex model expressed in Equation 15. Thus, the measurement model of the curve-of-factors model and the multiple indicator quasi-simplex model are identical when the number of indicators and observation time points are the same.

*Structural model*. The second-order portion of the structural model specifies the growth parameters, i.e., the level and the shape, of the first-order factors. Because the first-order factors are modeled in terms of second-order factors, the equation for this structural part of the model is slightly more complex than that expressed in Equation 2. The structural portion of the curve-of-factors model can be expressed as (Hancock et al., 2001):

$$\boldsymbol{\eta} = \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}, \tag{20}$$

where $\boldsymbol{\eta}$ is a vector of the $\eta_{ij}$ first-order factor scores, $\boldsymbol{\Gamma}$ is a matrix of second-order factor loadings reflecting the growth pattern underlying the $\eta_j$ factors, $\boldsymbol{\xi}$ is a vector of exogenous latent factors capturing the level (i.e., $\alpha$) and shape (i.e., $\beta$) parameters of the latent variable, and $\boldsymbol{\zeta}$ is a vector of random normal disturbances in the first-order factors. The expanded matrix form of this equation for an unconditional linear curve-of-factors model would be:

$$\begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \vdots \\ \eta_{ij} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & j \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} + \begin{bmatrix} \zeta_{i1} \\ \zeta_{i2} \\ \vdots \\ \zeta_{ij} \end{bmatrix} \tag{21}$$

Similar to the loadings of the latent growth model, the loadings of the latent growth part of the curve-of-factors model (e.g., the $\mathbf{\Gamma}$ matrix) can be fixed to values that reflect a specific hypothesis about the shape of the growth. As modeled in Equation 21, the intercept value is the average value at the first measurement occasion. As stated earlier, the initial value of interest is not restricted to the first measurement occasion; it can be modeled to occur at any time point. Fixing one loading of the shape parameter equal to zero defines the corresponding measurement time as a reference for the interpretation of the latent means of the level and shape parameters. The loadings of the latent growth parameters can also be estimated freely using the data rather than being specified by the researcher.

The second-order latent growth parameters (i.e., the level and shape) are modeled as:

$$\boldsymbol{\xi} = \boldsymbol{\mu} + \boldsymbol{\zeta}, \tag{22}$$

where $\boldsymbol{\xi}$ is a vector containing the level and shape parameters for each individual, $i$, $\boldsymbol{\mu}$ is the vector of latent means $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\beta$ of the level and shape respectively for all individuals, and $\boldsymbol{\zeta}$ is a vector of disturbances.

In expanded matrix format, this portion of the curve-of-factors model is:

$$\begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} = \begin{bmatrix} \mu_\alpha \\ \mu_\beta \end{bmatrix} + \begin{bmatrix} \zeta_{\alpha_i} \\ \zeta_{\beta_i} \end{bmatrix}. \tag{23}$$

*Covariance structure*. The second-order factor model increases the complexity of the implied covariance of this part of the model, also. The previous models described in this dissertation model the covariance of the measurement model and the covariance of the structural model. The curve-of-factors model models the covariance of the measurement model, the covariance of the first-order structural model, and the covariance of the second-order structural model.

The variance-covariance equation for the common factor portion of the model can be expressed as

$$\mathbf{\Sigma_{yy}} = \mathbf{\Lambda_y \Omega \Lambda'_y} + \mathbf{\Theta_\varepsilon}, \tag{24}$$

where $\mathbf{\Sigma_{yy}}$ is the variance-covariance matrix of the $y_{kj}$ indicators, $\mathbf{\Lambda_y}$ is the matrix containing the factor loadings of the items on the latent factors, $\mathbf{\Omega}$ is the variance-covariance matrix of the $\eta_j$ first-order latent factors, and $\mathbf{\Theta}_\varepsilon$ is a variance-covariance matrix of the $\varepsilon_{kj}$ measurement errors of the items (Leite, 2007).

The measurement errors of the items are often assumed to be uncorrelated, making the covariance structure of the measurement model equal to:

$$\mathbf{\Theta} = \begin{bmatrix} \sigma_{1j}^2 & 0 & 0 & 0 \\ 0 & \sigma_{2j}^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_{kj}^2 \end{bmatrix} \tag{25}$$

If the first-order factors are assumed to be uncorrelated as is common in the curve-of-factors model, the covariance structure of the first-order latent factor portion of the model is equal to:

$$\Psi = \begin{bmatrix} \sigma_{\eta_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{\eta_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{\eta j}^2 \end{bmatrix}. \tag{26}$$

The variances and the covariance of the disturbances of the second-order factors – the growth parameters – are also estimated because they correspond to the variances and covariance of the level and shape factors. The covariance structure of the linear growth parameters is equal to:

$$\mathbf{\Phi} = \begin{bmatrix} \sigma_{\zeta_\alpha}^2 & \sigma_{\zeta_\alpha \zeta_\beta} \\ \sigma_{\zeta_\alpha \zeta_\beta} & \sigma_{\zeta_\beta}^2 \end{bmatrix}, \tag{27}$$

where $\sigma_{\zeta_\alpha}^2$ is the variance of the level factor, $\sigma_{\zeta_\beta}^2$ is the variance of the shape factor, and $\sigma_{\zeta_\alpha \zeta_\beta}$ is the covariance between the level and shape.

The implied covariance matrix for the latent-growth portion of the model is thus,

$$\mathbf{\Omega} = \mathbf{\Gamma_\eta \Phi \Gamma'_\eta + \Psi}, \tag{28}$$

where the terms are as defined above. Substitution of the implied covariance matrix for the latent-growth portion of the model (i.e., Equation 28) into Equation 24, the implied covariance matrix for the first-order common factor portion of the model, gives the model implied covariance matrix,

$$\boldsymbol{\Sigma}_{\mathbf{yy}} = \boldsymbol{\Lambda}_{\mathbf{y}}\left(\boldsymbol{\Gamma}_{\boldsymbol{\eta}}\boldsymbol{\Phi}\boldsymbol{\Gamma}'_{\boldsymbol{\eta}} + \boldsymbol{\Psi}\right)\boldsymbol{\Lambda}'_{\mathbf{y}} + \boldsymbol{\Theta}_{\boldsymbol{\varepsilon}} \tag{29}$$

An unconditional curve-of-factors growth model depicting the growth of latent constructs is presented in Figure 4.

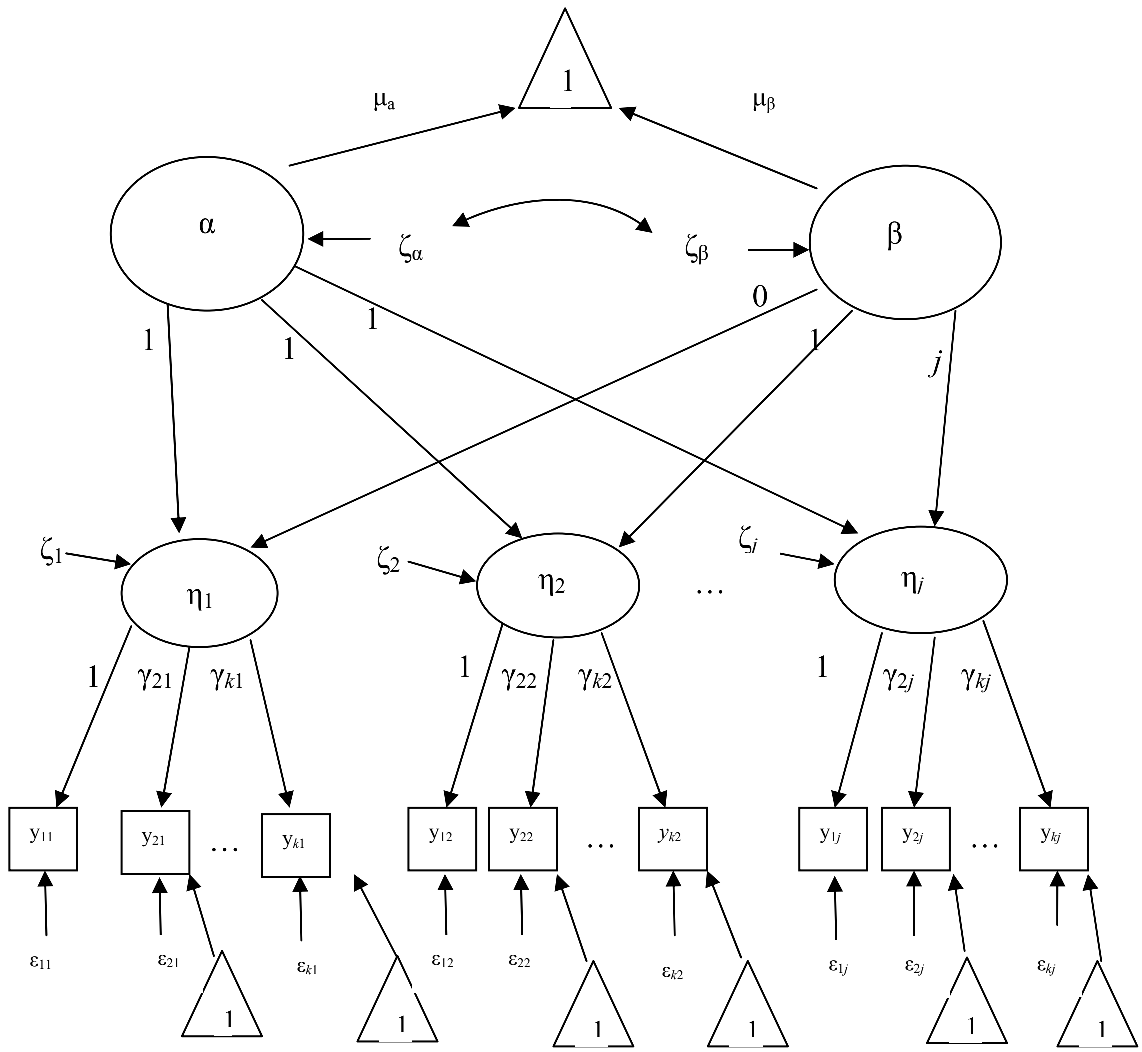


Figure 4. Unconditional curve-of-factors model.

*Serial correlation in the curve-of-factors model*. As mentioned earlier, time series data can be modeled for two stochastic processes: autoregressive and moving average (Box & Jenkins, 1976). The added complexity of the curve-of-factors model implies that these two processes can be modeled in separate potions of the model, i.e., at the structural level and/or at the measurement level. This section of the dissertation will illustrate how a curve-of-factors model can incorporate an ARMA(1, 1) process in the first-order factors as well as in the measurement error structure. Both models are constructed so that the current value of the time series is modeled as a function of previous values.

Adding an ARMA(1, 1) process to the structural model of the curve-of-factors model would incorporate time series processes as described in the section that illustrated the quasi-simplex model. The first-order structural model equation for the curve-of-factors model can be presented as,

$$\boldsymbol{\eta}_{\mathbf{t}} = \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\beta}_{\mathbf{21}}\boldsymbol{\eta}_{\mathbf{t-1}} + \boldsymbol{\xi}_{\mathbf{t}} + \boldsymbol{\Psi}_{\mathbf{21}}\boldsymbol{\xi}_{\mathbf{t-1}}, \tag{30}$$

which is a combination of Equations 16 and 20. The expanded matrix form of this portion of the model for five time points would be,

$$\begin{bmatrix}\eta_{i1}\\ \eta_{i2}\\ \eta_{i3}\\ \eta_{i4}\\ \eta_{i5}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 1\\ 1 & 2\\ 1 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}\alpha_i\\ \beta_i\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 & 0 & 0\\ \beta_{21} & 0 & 0 & 0 & 0\\ 0 & \beta_{21} & 0 & 0 & 0\\ 0 & 0 & \beta_{21} & 0 & 0\\ 0 & 0 & 0 & \beta_{21} & 0\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{bmatrix} + \begin{bmatrix}1.0 & 0 & 0 & 0 & 0\\ \gamma_{67} & 1.0 & 0 & 0 & 0\\ 0 & \gamma_{67} & 1.0 & 0 & 0\\ 0 & 0 & \gamma_{67} & 1.0 & 0\\ 0 & 0 & 0 & \gamma_{67} & 1.0\end{bmatrix}\begin{bmatrix}\eta_6\\ \eta_7\\ \eta_8\\ \eta_9\\ \eta_{10}\end{bmatrix} + \begin{bmatrix}\zeta_{i1}\\ \zeta_{i2}\\ \zeta_{i3}\\ \zeta_{i4}\\ \zeta_{i5}\end{bmatrix} \tag{31}$$

The model implied covariance structure is presented in Equation 29. The $\boldsymbol{\Psi}$ matrix can be expressed with two additional terms to account for the serial correlation parameters, as presented below (SAS Institute, 2005).

$$\sigma_\eta^{2*}\begin{bmatrix}1 & \gamma & \rho\gamma & \rho^2\gamma & \rho^3\gamma\\ \gamma & 1 & \gamma & \rho\gamma & \rho^2\gamma\\ \rho\gamma & \gamma & 1 & \gamma & \rho\gamma\\ \rho^2\gamma & \rho\gamma & \gamma & 1 & \gamma\\ \rho^3\gamma & \rho^2\gamma & \rho\gamma & \gamma & 1\end{bmatrix}, \tag{32}$$

where $\sigma_{\eta}^{2*}$ and $\gamma$ are functions of the autoregressive $(\rho)$ and moving average parameters.[1] The curve-of-factors model with an ARMA(1, 1) process integrated within the structural model is presented in Figure 5.

[1] $\sigma^{2*} = \frac{\sigma_{\eta}^{2}(1+\theta^{2}-2\rho\theta)}{(1-\rho^{2})}$, where $\rho$ is the autoregressive parameter $(\beta_{21})$ and $\theta$ is the moving average parameter. It can be shown that $\gamma = \frac{-\theta}{1+\theta^{2}}$ (Box & Jenkins, 1976).

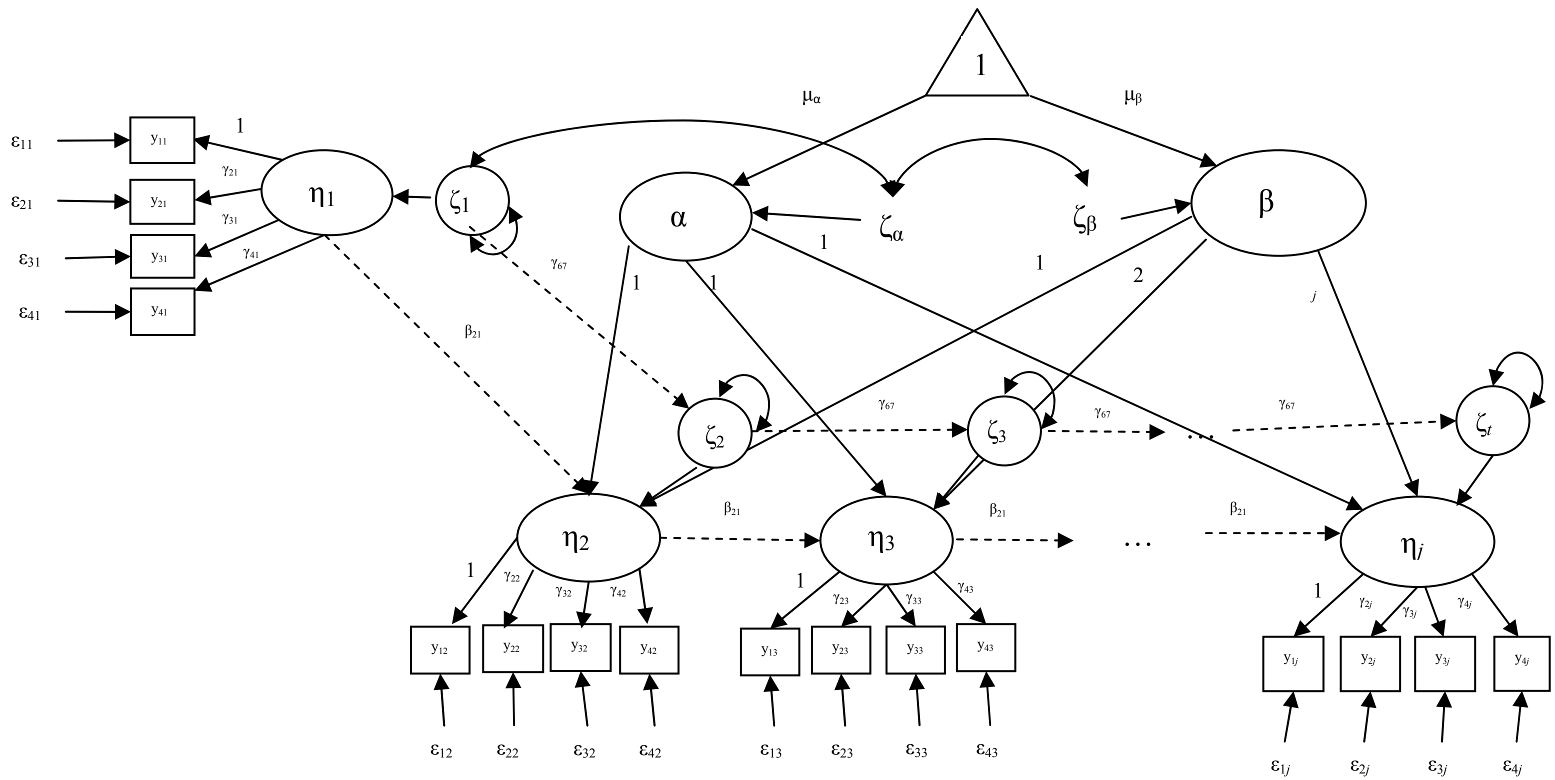


Figure 5. ARMA(1, 1) curve-of-factors model.

Adding an ARMA(1, 1) processes to the measurement model of the curve-of-factors would involve multiple equations of the form,

$$y_t = \tau + y_t\eta + \beta y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}\ , \tag{33}$$

where each $y_t$ manifest variable is a function of its associated latent variable η, a correlation with the previously measured manifest variable defined by an autoregressive process ($\beta$), a correlation with the measurement error of the previously measured manifest variable defined by a moving average process ($\theta$), and independent error ($\varepsilon_t$).

The implied covariance structure of a curve-of-factors model where serial correlation is present in the measurement model is described in Equation 30, and its expanded form is presented in Equation 31 with one important exception; the $\mathbf{\Theta}_\varepsilon$ matrix of measurement errors would be:

$$\mathbf{\Theta}_{\boldsymbol{\varepsilon}} = \sigma^{2*}\begin{bmatrix} 1 & 0 & 0 & \gamma & 0 & 0 & \rho\gamma & 0 & 0 \\ 0 & 1 & 0 & 0 & \gamma & 0 & 0 & \rho\gamma & 0 \\ 0 & 0 & 1 & 0 & 0 & \gamma & 0 & 0 & \rho\gamma \\ \gamma & 0 & 0 & 1 & 0 & 0 & \gamma & 0 & 0 \\ 0 & \gamma & 0 & 0 & 1 & 0 & 0 & \gamma & 0 \\ 0 & 0 & \gamma & 0 & 0 & 1 & 0 & 0 & \gamma \\ \rho\gamma & 0 & 0 & \gamma & 0 & 0 & 1 & 0 & 0 \\ 0 & \rho\gamma & 0 & 0 & \gamma & 0 & 0 & 1 & 0 \\ 0 & 0 & \rho\gamma & 0 & 0 & \gamma & 0 & 0 & 1 \end{bmatrix} \tag{34}$$

where the autoregressive ($\rho$), moving average ($\gamma$), and variance ($\sigma^{2*}$) terms are as described previously. Mismodeling the measurement error covariance structure has been found to bias the level-two variance/covariance estimates in multilevel growth curve models (Ferron et al., 2002; Kwok et al., 2007 Murphy & Pituch, 2009), the type I error rates of tests of the fixed effects in multilevel growth curve models (Ferron et al. 2002; Kwok et al., 2007; Murphy & Pituch, 2009), and the standard errors of the fixed effects in multilevel growth curve models (Kwok et al.). Further, mismodeling the measurement error structure of latent growth models has been found to bias estimates of the fixed effects and variance components (Sivo et al., 2005). To date no one has studied the effect of unmodeled or mismodeled covariance structures in the curve-of-factors model. The purpose of this study was to examine the performance of the curve-of-factors model when serial correlation was present in the structural model. In addition, when serial correlation was present in the structural model, the performance of new models that combine aspects of quasi-simplex and latent growth models was examined.

# Chapter III: Method

In this study, the effects of serial correlation on the growth parameters of the curve-of-factors model were evaluated. Monte Carlo methods were used to generate 1,000 samples for each condition. The conditions were designed to be as similar as possible to situations that may be encountered in applied studies, where researchers must make decisions regarding appropriate sample sizes, series lengths and model specifications. In this section, the conditions and parameters are described, followed by the data generation and analysis procedures, and finally the methods that were used to evaluate the results.

### *Conditions and Parameters*

This simulation study modifies a SAS macro developed by Fan, Felsöványi, Sivo, and Keenan (2001). Monte Carlo methods were used to generate longitudinal data with a single set of growth parameters data using SAS/IML. The data were then analyzed using the SAS PROC CALIS procedure. The first-order factors of the curve-of-factors model used to generate data for this study were indicated by four observed variables that were measured repeatedly across equally spaced time points as described in Equation 1.

The second-order factors modeled the level and shape of the first-order latent factors as described in Equation 20. For each condition, the population means of the level and shape were set to 0 and 0.5 respectively. Consistent with the parameter values used in previous simulation studies (e.g., Leite, 2007; Sivo et al., 2005), the variance of the level parameter was set to 0.5, and the variance of the shape parameter was set to 0.1. The covariance between the level and shape was set to 0. A linear growth trajectory was specified for all conditions, with the intercept modeled as the first measurement occasion.

This was accomplished by setting the shape factor loading of the first measurement occasion to zero, with subsequent factor loadings increasing in increments of one. The level factor loadings all were set to one.

The parameters of the measurement model (i.e., the observed indicators) were simulated to be identical across time, meaning strict factorial invariance was generated. The item intercepts all were generated to be 0, the factor loadings all were generated to be 1, and the error variances for all items were generated to be 1. Note that substitution of the values of the items' factor loadings and the error variances into the formula (Deshon, 1998; Ruetenberg & Gustafson, 1992)

$$\rho_{cc'} = \frac{\left(\sum_{k=1}^{K} \lambda_k\right)^2}{\left(\sum_{k=1}^{K} \lambda_k\right)^2 + \sum_{k=1}^{K} \sigma_{\varepsilon_k}^2}, \tag{35}$$

where $\lambda_k$ represents an item's factor loading and $k$ represents the number of items per factor, results in a reliability coefficient of .8.

Four factors were systematically varied in this study. First, the values and parameters of the autocorrelation process were generated to model either an ARMA(1, 1), AR(1), or control (i.e., no serial correlation) process. Second, the sample sizes were simulated to be 100, 200, 500, or 1,000. Third, the measurement occasion series length was varied as either 5 or 8 simulated measurement occasions. Fourth, the curve-of-factors model utilized to analyze the data was varied in three ways: 1) a curve-of-factors model without serial correlation parameters was specified; 2) an AR(1) parameter was added to the first-order structural portion of the model; and 3) two ARMA(1, 1) parameters were added to the first-order structural portion of the model. Each of these four factors is described in more detail below.

Note that serial correlation can be present in the structural model, the measurement model, or both portions of the curve-of-factors model. This dissertation will focus on serial correlation in the structural model for two reasons: the first is that the curve-of-factors model appears to be well suited to combining features of the latent growth and quasi-simplex models, which may provide a means of modeling both serial correlation and growth processes from the same data set. The second reason this dissertation will focus on serial correlation in the structural model is that the measurement model of the curve-of-factors model is not well suited to modeling serial correlation. For example, four indicators per latent variable measured repeatedly could potentially result in four different autocorrelation functions, which would be difficult to model. Although this dissertation will not study the effect of serial correlation in the measurement model of the curve-of-factors model, it would seem to be a topic that is worthy of future exploration.

*Serial Correlation*

For the ARMA(1, 1) process, the autocorrelations of the AR(1) portion of the process ($\varphi$) will take on two different values, one correlation with moderate magnitude, .5, and one correlation with a large magnitude, .8. In addition, the correlations of the MA(1) process ($\theta$) also will take on two values; a correlation .3 was paired with the large magnitude correlation of the AR(1) process (i.e., .8), and a correlation of -.3 was paired with the moderate magnitude correlation of the AR(1) process (i.e., .5). These ARMA(1, 1) parameter values were selected and paired because in combination they model serial correlation that decays more slowly and more quickly respectively than a pure AR(1) process[2].

By setting θ = 0, two AR(1) processes were generated, a large autocorrelation where φ = .8 and a moderate autocorrelation where φ = .3. Finally, by constraining both φ and θ to equal zero, a set of data without serial correlation was generated as a control model. All parameter values are within the ranges of values commonly studied in previous simulations of AR(1) and ARMA(1, 1) data (Ferron et al., 2002; Hamaker, Dolan, & Molenaar, 2002; Murphy & Pituch, 2009; Sivo et al., 2005; Sivo & Willson, 2000).

*Sample Size*

Consistent with Leite's (2007) previous curve-of-factors simulation study, four sample sizes were simulated: two small (100 and 200), and two large (500 and 1000). These sample sizes are also within ranges used in simulation studies examining the performance of latent growth models and time series models (Hamaker, Dolan, & Molenaar, 2002; Sivo et al., 2005; Sivo & Willson, 2000).

*Series Length*

A third factor that was systematically varied in generating the data for this study is series length, with two series lengths implemented. Sivo et al. (2005) recommend at least 5 measurement occasions when specifying an ARMA(1, 1) stochastic model a priori. Therefore, the smallest series length used was 5. The other series length examined was based on previous simulation studies involving serial correlation (Ferron et al., 2002;

---

[2] The rate of decay of the ARMA (1, 1) process is determined by the equations, $\rho_1 = \frac{(1-\phi_1\theta_1)(\phi_1-\theta_1)}{1+\theta_1^2-2\phi_1\theta_1}$, and $\rho_2 = \phi_1\rho_1$ (Box & Jenkins, 1976), where ϕ is the AR coefficient and Θ is the MA coefficient.

Hamaker, Dolan, & Molenaar, 2002; Sivo et al., 2005; Sivo & Willson, 2000), where the most commonly specified series length was 8.

*Model Specification*

A fourth factor, the specification of the curve-of-factors model, was varied in the analysis of the generated data sets. This fourth factor, a repeated measures design factor, consisted of three levels: a pure curve-of-factors model, a curve-of-factors model integrated with an AR(1) autoregressive parameter in the structural model, and a curve-of-factors model integrated with two ARMA(1, 1) parameters in the structural model. This fourth factor was crossed with all study design factors. Thus, the study design had 120 cells. To summarize, the data were generated as a 5 (autocorrelation structure) x 4 (sample size) x 2 (series length) x 3 (model specification) factorial design. For each cell, a total of 1,000 data sets were generated resulting in a total of 120,000 data sets.

*Data Generation*

First, the matrix equations for the covariance parameters of the curve-of-factors model were filled with the population values of the growth model for each condition. Next, the matrix equations were computed using SAS IML programming language (SAS Institute, 2005) to obtain the implied population covariance matrix, from which a population correlation matrix and matrix of standard deviations were derived. Next, for each condition SAS PROC FACTOR generated a factor pattern matrix based on the population correlation matrix of the variables (i.e., the simulated observed indicators).

Because the observed variables were generated to have intercepts of zero, no mean structure was specified. The factor pattern matrix was used to generate variables from which random variable data matrices in turn were generated. The growth parameters were added to the initial set of variables generated from the factor pattern matrix prior to the generation of the random variable data matrices.

The three curve-of-factors models were fit to the 1,000 data sets for each of the conditions, and the convergence rates and percentages of inadmissible solutions were recorded. The initial sets were analyzed without removing the inadmissible solutions. Next, inadmissible solutions were removed and additional data sets were simulated until 1,000 admissible solutions were obtained for each condition within each method. The reason for obtaining 1,000 sets of results for each condition with each method is that the models were compared across replications. Thus, the means and standard deviations of the parameter estimates and fit criteria should be obtained from the same number of observations.

The SAS data sets for each condition were saved so they could be used in the data analysis stage as a method of model comparison.

### *Data Analysis*

The first step in comparing the performance of the three growth models was an examination of the convergence rates. The percentage of non-convergent cases and non-admissible solutions for the first 1,000 datasets generated under each method under each condition was reported.

To determine how well parameters were estimated under the different model specifications, bias was computed for the point estimates and standard error estimates of the fixed effects (i.e., the level and shape). Bias was also computed for point estimates of the random effects (i.e., the variance of the level, the variance of the shape, and the covariance between the level and the shape). Relative bias was calculated using the equation,

$$RB = \frac{\bar{\hat{\theta}} - \theta}{\theta}, \tag{36}$$

where $\bar{\hat{\theta}}$ is the mean of the parameter estimates across 1,000 replications and $\theta$ is the true parameter value. The parameter estimates were considered biased when the mean absolute value of relative bias (MARB) exceeded .05 (Hoogland & Boomsma, 1998). When the true parameter value is 0, relative bias cannot be calculated. Under such circumstances, I considered estimates to be biased when the mean absolute value of the simple bias across 1,000 replications exceeded .05 (i.e., when the mean point estimate was outside the range between -.05 and .05).

The relative bias of the standard errors was defined as

$$B\left[\widehat{se}\left(\hat{\theta}\right)\right] = \frac{\overline{\widehat{se}}(\hat{\theta}) - SD(\hat{\theta})}{SD(\hat{\theta})} \tag{37}$$

where $\overline{\widehat{se}}\left(\hat{\theta}\right)$ is the mean of the estimated standard errors and $SD\left(\hat{\theta}\right)$ is the standard deviation of the parameter estimate, both calculated across 1,000 replications. The standard error estimates were considered to be acceptable if the MARB was less than 0.1 (Hoogland & Boomsma, 1998).

When modeling data relationships using SEM, a fundamental issue that must be determined prior to interpreting the parameter estimates is whether or not the model fits the data. Model fit refers to the degree to which the model-implied covariance matrix matches the observed covariance matrix of the variables (Bollen, 1989). In accordance with Leite's (2007) simulation study examining the performance of the curve-of-factors model, the models in this dissertation were evaluated based on chi-square statistics and overall fit criteria that indicated acceptable fit.

The three overall fit criteria examined were the comparative fit index (CFI), the Tucker-Lewis Index (TLI), and the root mean squared error of approximation (RMSEA). The goodness of fit (GOF) for each model specification across each condition was evaluated according to the proportion of times the CFI, TLI, and RMSEA indicated acceptable fit in accordance with the recommendations of Hu and Bentler (1999). According to these recommendations, models can be considered to fit the data well if they produce values greater than or equal to .95 for the CFI and TLI, and values less than or equal to .05 for the RMSEA. The GOF proportion was calculated using an indicator variable, where the model was given a 1 if a particular fit statistic (i.e., chi-square, CFI, TLI, and RMSEA) indicated that the model fit the data well in accordance with Hu and Bentler's recommendations, and a 0 otherwise. I also examined the relative bias of the chi-square statistic as an indicator of GOF.

Evaluating the GOF using chi-square statistics was done in two ways; the first was a direct interpretation of the chi-square statistic itself whereupon a p-value > .05 indicated that the model fit the data well (i.e., GOF could not be rejected). The second method of evaluating model fit using the chi-square statistic involved calculating its relative bias according to the equation,

$$B(\chi^2_{rc}) = \frac{\chi^2_{rc} - E[\chi^2_c]}{E[\chi^2_c]}, \tag{38}$$

where $\chi^2_{rc}$ is the chi-square statistic at replication $r$ of condition $c$ and $E[\chi^2_c]$ is the expected chi-square of condition $c$. For a correctly specified model the expected chi-square statistic is equal to the degrees of freedom of the model.

The reason that the relative bias of the chi-square statistic was evaluated in addition to the direct interpretation of the chi-square statistic is that when sample sizes are large, the chi-square statistic is known to be sensitive to very small degrees of misfit. This is because the chi-square statistic is an indicator of exact fit. The relative bias of the chi-square statistic can be considered acceptable if its magnitude is less than .05 (Hamilton, Gagné, & Hancock, 2003; Leite, 2007). Use of the relative bias of the chi-square statistics also enabled comparisons to be made across model specifications using multivariate analysis of variance (MANOVA).

Following the recommendation of Hauck and Anderson (1984) that simulation studies be analyzed using the same tools as other experimental studies, factorial ANOVAs and MANOVAs were conducted using sample size, series length, autoregressive parameter, and moving average parameter as explanatory variables. In sections where the results of a single model are presented, ANOVAs were conducted as a means of analyzing the effects of the study design conditions on each model separately. When comparing the results across models, the model specification was treated as a repeated measures factor. For all analyses the outcome measures of interest were: the simple bias of the estimate of the mean of the level, the simple bias of the covariance of the level and shape, the relative bias of the mean of the shape, the relative bias of the variance of the level, the relative bias of the variance of the shape, and the relative bias of the chi-square statistic.

Due to the large number of observations, standard probability values were not considered to be indicative of practical significance. Therefore, effect sizes were calculated using the formula $\eta^2 = \frac{SS_{effect}}{SS_{effect} + SS_{error}}$, which is known as partial $\eta^2$. Thus, all reported values of $\eta^2$ in this dissertation are partial $\eta^2$. Effect size values greater than .01 were considered to be practically significant. The .01 value was chosen based on previous research (e.g., Krull & MacKinnon, 1999), as well as because .01 is the threshold value at which $\eta^2$ is considered to be a small effect. This method of calculating $\eta^2$ is used to estimate effect sizes independent of other effects included in the model. Because these effect size estimates can sum to values greater than 1, an estimate cannot be interpreted as the proportion of total variance explained.

# Chapter IV: Results

This chapter presents the results of the simulation study designed to compare the performance of three curve-of-factors models when serial correlation and growth processes were present in the same data set. The chapter is divided into four sections. In the first three sections, the results of the curve-of-factors model, the curve-of-factors model fitted with an AR(1) parameter, and the curve of factors model fitted with two ARMA(1, 1) parameters are presented. The fourth section compares results across the three different models.

I evaluated the results for the three models with respect to convergence rates, relative bias of the estimates of the means, standard errors, and variances and covariance of the level and shape. In addition, a GOF proportion was calculated for each model across each condition. The purpose behind the GOF analysis was to examine the sensitivity of the fit criteria to model fit under conditions where the estimating model was specified correctly as well as conditions where the estimating model was misspecified.

The results of the ANOVAs and MANOVAs for measures of relative bias within and between models are presented for conditions in which the $\eta^2$ effect sizes were found to be larger than 0.01. In this chapter I indicate whether the relative bias for each estimated parameter was considered to be acceptable under the conditions for each model, and whether the magnitude of the bias was influenced by a particular study condition or combination of study conditions.

## *Curve-of-factors Model*

### *Convergence and Proportion of Inadmissible Solutions*

The convergence rate for the curve-of-factors model across all conditions was 100%. Although non-convergence was not a problem, there were solutions that were not admissible. Specifically, the estimated variance\covariance matrices were non-positive definite across conditions due to negative estimates of the level and shape variances. Because the maximum likelihood estimator used in the SAS PROC CALIS software produces unbounded estimates (Wothke, 1993), the estimated variances may be inadmissible (i.e., negative). The curve-of-factors model had five inadmissible solutions when the sample size was 100, the series length was 5 and there was no autocorrelation present in the data. Otherwise, analysis of the data using the curve-of-factors model resulted in admissible solutions across all conditions.

### *Relative and Simple Bias*

This section of the paper discusses the relative bias of estimates of the means, standard errors, and variances and covariance of the level and shape. The definition of an unbiased estimator is one in which the expected value of the distribution of estimates equals the parameter value (i.e., $E(\hat{\theta}) = \theta$) (Wackerly, Mendenhall, & Scheaffer, 1996). The calculation of $E(\hat{\theta})$ therefore requires the full distribution of parameter estimates. Removal of inadmissible solutions truncates the distribution of variance estimates at zero and potentially alters the distribution of the other parameter estimates. Therefore, the relative bias of all parameter estimates was calculated twice: once before inadmissible solutions were removed from the data set, and once after the inadmissible solutions were removed and replaced with additional admissible solutions.

In this study, analysis of the data using the curve-of-factors model resulted in a total of 5 inadmissible solutions. The inadmissible solutions were removed from the data set and extra replications were generated until a total of 1,000 admissible solutions were contained for each condition. Note that replacing the 5 inadmissible solutions for the curve-of-factors model did not noticeably change the relative bias of any of the estimates of interest, so only one set (i.e., the complete set) of parameter estimates is presented.

As depicted in Tables 1 and 2, the point estimates of the fixed effects were within acceptable levels of bias under the curve-of-factors model across all of the examined conditions.

Table 1.

*Mean Simple Bias of the Level Parameter under the Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 0.003 | -0.004 | -0.006 | 0.003 | 0.007 |
| | 200 | -0.004 | < 0.001 | 0.003 | 0.005 | -0.003 |
| | 500 | 0.001 | -0.001 | -0.004 | 0.001 | -0.001 |
| | 1000 | < 0.001 | 0.001 | -0.002 | 0.001 | 0.001 |
| 8 | 100 | 0.001 | 0.003 | -0.001 | -0.002 | -0.002 |
| | 200 | < 0.001 | -0.004 | -0.002 | 0.003 | 0.006 |
| | 500 | 0.001 | -0.003 | < 0.001 | < 0.001 | -0.001 |
| | 1000 | -0.001 | < 0.001 | 0.002 | -0.002 | < 0.001 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Table 2.

*Mean Relative Bias of the Shape Parameter under the Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | < 0.001 | < 0.001 | 0.006 | -0.004 | -0.003 |
| | 200 | 0.004 | 0.002 | 0.002 | 0.001 | 0.002 |
| | 500 | < 0.001 | 0.001 | 0.001 | < 0.001 | -0.003 |
| | 1000 | -0.001 | 0.001 | 0.001 | -0.002 | < 0.001 |
| 8 | 100 | -0.001 | < 0.001 | -0.001 | -0.007 | -0.003 |
| | 200 | < 0.001 | 0.001 | 0.001 | -0.009 | -0.002 |
| | 500 | 0.001 | 0.003 | -0.001 | < 0.001 | < 0.001 |
| | 1000 | 0.001 | < 0.001 | 0.000 | -0.002 | 0.001 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

ANOVA results indicated that the mean simple bias of the covariance of the level and shape depended on the magnitude of the serial correlation present in the data ($\eta^2$ = .74). There was also an interaction between series length and the magnitude of the serial correlation present in the data ($\eta^2$ = .35). When no serial correlation was present in the data, estimates of the covariance of the level and shape were unbiased across sample size and series length conditions (see Table 3). In contrast, the covariance of the level and shape was underestimated across all conditions when serial correlation was present in the data. Under non-zero serial correlation conditions, the mean simple bias ranged from -0.058 under the AR(1) condition where φ = .3, the sample size was 1,000, and the series length was 8 to -0.340 under the ARMA(1, 1) condition where φ = .5 and θ = -.3, the sample size was 100, and the series length was 5 (see Table 3). Note also that the mean absolute value of the simple bias decreased as series length increased under the AR(1) condition where φ = .3 and the ARMA(1, 1) condition where φ = .5 and θ = -.3, but the mean absolute value of the simple bias remained constant or increased as series length increased under the other two non-zero serial correlation conditions.

Table 3.

*Mean Simple Bias of the Covariance of the Level and Shape Parameter under the Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | < 0.001 | **-0.103** | **-0.141** | **-0.158** | **-0.340** |
| | 200 | 0.004 | **-0.106** | **-0.139** | **-0.153** | **-0.332** |
| | 500 | 0.002 | **-0.105** | **-0.140** | **-0.155** | **-0.341** |
| | 1000 | 0.002 | **-0.105** | **-0.139** | **-0.153** | **-0.336** |
| 8 | 100 | 0.001 | **-0.057** | **-0.312** | **-0.161** | **-0.241** |
| | 200 | 0.002 | **-0.058** | **-0.310** | **-0.161** | **-0.243** |
| | 500 | 0.001 | **-0.057** | **-0.309** | **-0.155** | **-0.246** |
| | 1000 | < 0.001 | **-0.058** | **-0.310** | **-0.161** | **-0.244** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

The relative bias of the variance of the level under the curve-of-factors model depended on series length ($\eta^2 = .40$), magnitude of serial correlation in the data ($\eta^2 = .92$), and an interaction between the two factors ($\eta^2 = .67$). As with the estimates of the covariance of the level and the shape, estimates of the variance of the level were unbiased across the zero-autocorrelation conditions (see Table 4). In contrast with the estimates of the covariance of the level and shape, the variance of the level was overestimated across all conditions where serial correlation was present in the data.

When serial correlation was present in the data, the mean relative bias ranged from 0.651 under the AR(1) condition where $\varphi = .3$, the sample size was 1,000 and the series length was 8 to 5.389 under the AR(1) condition where $\varphi = .8$, the sample size was 100, and the series length was 8. As was seen for estimation of the covariance parameter, the estimation improved as series length increased under two autocorrelation specifications (i.e., AR(1) where $\varphi = .3$ and ARMA(1, 1) where $\varphi = .5$ and $\theta = -.3$), but estimation worsened as series length increased under the other two autocorrelation specifications (see Table 4).

Table 4.

*Mean Relative Bias of the Variance of the Level under the Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | -0.004 | **0.817** | **2.257** | **2.436** | **3.045** |
| | 200 | 0.000 | **0.817** | **2.244** | **2.434** | **3.005** |
| | 500 | -0.001 | **0.802** | **2.250** | **2.437** | **3.059** |
| | 1000 | -0.007 | **0.818** | **2.259** | **2.433** | **3.022** |
| 8 | 100 | 0.001 | **0.652** | **5.389** | **3.186** | **2.860** |
| | 200 | -0.003 | **0.653** | **5.387** | **3.232** | **2.866** |
| | 500 | -0.015 | **0.657** | **5.361** | **3.203** | **2.879** |
| | 1000 | -0.001 | **0.651** | **5.374** | **3.223** | **2.870** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

The magnitude of bias in the estimates of the shape variance also depended on series length ($\eta^2$ = .62), autocorrelation ($\eta^2$ = .92), and an interaction between the two factors ($\eta^2$ = .92). Similar to the estimates of the variance of the level parameter, the estimates of the variance of the shape parameter were unbiased when no serial correlation was present in the data, but they were overestimated across all other conditions (see Table 5). The interaction between series length and serial correlation repeated the trend where estimation improved as series length increased for the AR(1) process where φ = .3 and the ARMA(1, 1) process where φ = .5 and θ = -.3 but worsened under the other two serial correlation processes as series length increased. The magnitude of the bias was greatest under the ARMA(1, 1) condition where φ = .8 and θ = .3 and the series length was large, whereupon the variance was overestimated by more than 6 times the nominal variance (see Table 5).

Table 5.

*Mean Relative Bias of the Variance of the Shape under the Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 0.002 | **0.526** | **0.707** | **0.772** | **1.743** |
| | 200 | -0.008 | **0.523** | **0.696** | **0.754** | **1.724** |
| | 500 | -0.004 | **0.524** | **0.705** | **0.775** | **1.737** |
| | 1000 | -0.010 | **0.522** | **0.708** | **0.764** | **1.723** |
| 8 | 100 | -0.013 | **0.157** | **0.902** | **6.351** | **0.700** |
| | 200 | -0.002 | **0.161** | **0.892** | **6.339** | **0.701** |
| | 500 | -0.001 | **0.167** | **0.886** | **6.351** | **0.721** |
| | 1000 | 0.001 | **0.168** | **0.893** | **6.353** | **0.710** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

The MARB of estimates of the standard errors of the level and shape are presented in Tables 6 and 7. As can be seen in Tables 6 and 7, the standard errors of the fixed effects under the curve-of-factors model were considered to be within acceptable limits of bias across all conditions.

Table 6.

*Mean Relative Bias of the Standard Error of the Level under the Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
|---|---|---|---|---|---|---|
| 5 | 100 | -0.032 | -0.010 | -0.028 | -0.044 | -0.021 |
| | 200 | -0.044 | 0.034 | -0.029 | 0.013 | -0.048 |
| | 500 | -0.031 | 0.067 | 0.003 | 0.082 | -0.010 |
| | 1000 | -0.033 | -0.025 | -0.009 | 0.027 | -0.005 |
| 8 | 100 | -0.045 | -0.004 | -0.017 | -0.014 | -0.032 |
| | 200 | -0.018 | 0.006 | -0.025 | -0.057 | 0.001 |
| | 500 | 0.027 | -0.048 | -0.019 | -0.028 | 0.001 |
| | 1000 | -0.001 | 0.012 | -0.035 | 0.001 | 0.024 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Table 7.

*Mean Relative Bias of the Standard Error of the Shape under the Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
|---|---|---|---|---|---|---|
| 5 | 100 | -0.037 | -0.028 | -0.011 | -0.034 | -0.027 |
| | 200 | 0.002 | -0.001 | -0.001 | 0.012 | -0.075 |
| | 500 | -0.040 | 0.016 | 0.012 | -0.008 | 0.024 |
| | 1000 | 0.031 | -0.005 | -0.034 | 0.008 | -0.023 |
| 8 | 100 | -0.032 | -0.011 | -0.021 | -0.019 | -0.020 |
| | 200 | 0.008 | -0.004 | 0.001 | 0.050 | -0.024 |
| | 500 | 0.022 | 0.007 | -0.024 | 0.009 | 0.013 |
| | 1000 | 0.029 | 0.040 | -0.012 | -0.024 | 0.005 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

*Fit Criteria*

Inspection of Table 8 reveals that the proportion of times that the chi-square test indicated that the curve-of-factors model fit the data varied depending on sample size, series length, and autocorrelation magnitude. In the absence of autocorrelation (i.e., when the model was correctly specified), the GOF proportion ranged from 0.312 to 0.943. The highest GOF proportion occurred when the sample size was 1,000, and the lowest proportion occurred when the sample size was 100 and the series length was 8.

When serial correlation was present in the data (i.e., when the model was misspecified), the GOF proportion ranged from 0 to 0.825. In general, the GOF proportion decreased as sample size and series length increased under non-zero serial correlation conditions. The chi-square statistic was unlikely to indicate that the curve-of-factors model fit the data well under any condition when autocorrelation was present in the data in combination with large sample size and series length. GOF proportions were also noticeably lower under the ARMA(1, 1) condition where $\varphi = .5$ and $\theta = -.3$ (see Table 8).

Table 8.

*Proportion of Generated Data Sets that the Curve-of-Factors Model was indicated as fitting acceptably by the Chi Square Statistic while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 0.793 | 0.743 | 0.708 | 0.754 | 0.308 |
| | 200 | 0.888 | 0.789 | 0.745 | 0.825 | 0.112 |
| | 500 | 0.926 | 0.669 | 0.450 | 0.744 | 0.000 |
| | 1000 | 0.943 | 0.349 | 0.064 | 0.533 | 0.000 |
| 8 | 100 | 0.312 | 0.213 | 0.018 | 0.054 | 0.001 |
| | 200 | 0.775 | 0.409 | 0.000 | 0.056 | 0.000 |
| | 500 | 0.891 | 0.175 | 0.000 | 0.000 | 0.000 |
| | 1000 | 0.939 | 0.006 | 0.000 | 0.000 | 0.000 |

*Note*. Estimates based on 1,000 simulations.

The proportion of times that the TLI, and CFI indicated that the curve-of-factors fit the data was very high across all conditions as presented in Tables 9 - 10. The lowest GOF proportion for either of indexes was .948 (see Table 10). Under many conditions, the TLI and CFI indicated that the curve-of-factors model fit each of the 1,000 simulated data sets well (i.e., the proportion = 1.000), despite the fact that the variance components were estimated poorly under many of the conditions (e.g., ARMA(1, 1) where $\varphi = .8$ and $\theta = .3$).

The proportion of simulated data sets under which the curve-of-factors model was indicated as having adequate fit under the RMSEA criterion was also very high across most conditions. One exception was the combination of small sample size with the ARMA(1, 1) condition where $\varphi = .5$ and $\theta = -.3$, which had smaller GOF proportions. (see Table 11).

Table 9.

*Proportion of Generated Data Sets that the Curve-of-Factors Model was indicated as fitting acceptably by the Comparative Fit Index while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 0.999 | 1.000 | 0.999 | 1.000 | 0.996 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | 100 | 1.000 | 1.000 | 1.000 | 0.977 | 0.964 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

*Note*. Estimates based on 1,000 simulations.

Table 10.

*Proportion of Generated Data Sets that the Curve-of-Factors Model was indicated as fitting acceptably by the Tucker-Lewis Index while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) ($\varphi = .3$) | AR(1) ($\varphi = .8$) | ARMA(1, 1) ($\varphi = .8$, $\theta = .3$) | ARMA(1, 1) ($\varphi = .5$, $\theta = -.3$) |
| 5 | 100 | 0.999 | 1.000 | 0.999 | 1.000 | 0.995 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | 100 | 1.000 | 1.000 | 1.000 | 0.972 | 0.948 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

*Note*. Estimates based on 1,000 simulations.

Table 11.

*Proportion of Generated Data Sets that the Curve-of-Factors Model was indicated as fitting acceptably by the Root Mean Squared Error of Approximation while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 0.918 | 0.906 | 0.881 | 0.907 | 0.560 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 0.895 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | 100 | 0.945 | 0.872 | 0.427 | 0.653 | 0.124 |
| | 200 | 1.000 | 1.000 | 0.985 | 0.998 | 0.642 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 0.956 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 |

*Note*. Estimates based on 1,000 simulations.

As mentioned earlier in the paper the relative bias of the chi-square statistic was evaluated in addition to the direct interpretation of the chi-square statistic because the chi-square statistic is known to be sensitive to very small degrees of misfit. Use of the relative bias of the chi-square statistics also enabled comparisons to be made across conditions and model specifications using ANOVA. Under the curve-of-factors model, the relative bias of the chi-square statistic depended on sample size ($\eta^2$ = .78), series length ($\eta^2$ = .53), autocorrelation ($\eta^2$ = .86), and a 3-way interaction among the three factors ($\eta^2$ = .25). The chi-square statistic was within acceptable limits for 5 of the 8 conditions when no autocorrelation was present in the data, with bias occurring when the sample size was small (see Table 12).

By contrast, all chi-square statistics were positively biased when autocorrelation was present in the data. The magnitude of the bias increased as sample size and series length increased under the non-zero serial correlation conditions. The ARMA(1, 1) condition where resulted in the largest MARB across conditions (see Table 12).

Table 12.

*Mean Relative Bias of the Chi-square statistic under the Curve of Factors Model by Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | **0.085** | **0.101** | **0.115** | **0.102** | **0.232** |
| | 200 | 0.040 | **0.080** | **0.100** | **0.069** | **0.337** |
| | 500 | 0.017 | **0.124** | **0.191** | **0.098** | **0.751** |
| | 1000 | 0.005 | **0.223** | **0.354** | **0.163** | **1.492** |
| 8 | 100 | **0.137** | **0.167** | **0.266** | **0.220** | **0.340** |
| | 200 | **0.057** | **0.121** | **0.322** | **0.229** | **0.469** |
| | 500 | 0.025 | **0.178** | **0.679** | **0.445** | **1.053** |
| | 1000 | 0.010 | **0.323** | **1.330** | **0.855** | **2.074** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

*Curve-of-factors Model with AR(1) Parameter*

*Convergence and Proportion of Inadmissible Solutions*

The convergence rate for the AR(1) curve-of-factors model across all conditions was 100%. Although non-convergence was not a problem, there were solutions that were not admissible due to non-positive definite variance\covariance matrices. Analysis of the data using the AR(1) curve-of-factors model with a specified AR(1) parameter resulted in inadmissible solutions under every condition. The proportions of inadmissible solutions ranged from 0.003 under the ARMA(1, 1) serial condition where $\varphi = .5$ and $\theta = -.3$, the sample size was 1,000, and the series length was 5, to 0.504 when the sample size was 100 and the series length was 8 under the same autocorrelation condition. In general, increasing the sample size resulted in decreasing proportions of inadmissible solutions. By contrast, increasing the series length resulted in increasing proportions of inadmissible solutions across most conditions (see Table 13).

Table 13.

*Proportion of Inadmissible Solutions for the AR(1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 0.288 | 0.437 | 0.276 | 0.271 | 0.212 |
| | 200 | 0.220 | 0.374 | 0.274 | 0.155 | 0.108 |
| | 500 | 0.078 | 0.294 | 0.312 | 0.061 | 0.021 |
| | 1000 | 0.020 | 0.209 | 0.291 | 0.007 | 0.003 |
| 8 | 100 | 0.378 | 0.378 | 0.463 | 0.324 | 0.504 |
| | 200 | 0.159 | 0.248 | 0.405 | 0.171 | 0.367 |
| | 500 | 0.114 | 0.212 | 0.316 | 0.106 | 0.270 |
| | 1000 | 0.113 | 0.211 | 0.247 | 0.116 | 0.200 |

*Note*. Proportions based on 1,000 simulations.

*Relative and Simple Bias*

As mentioned earlier in the paper, the calculation of $E(\hat{\theta})$ requires a full distribution of parameter estimates (Wackerly et al., 1996). The inadmissible solutions from the AR(1) curve-of-factors model were removed from the data sets and extra replications were generated until a total of 1,000 admissible solutions were contained for each condition. Hence, the simple bias estimates were calculated twice in order to allow their comparison, once before inadmissible solutions were removed from the data and once after the inadmissible solutions were removed. Both sets of simple bias estimates are reported below.

Before the inadmissible solutions were removed from the data set, ANOVA results indicated that the simple bias of the level mean depended on series length ($\eta^2$ = .11), autocorrelation specification ($\eta^2$ = .56), and an interaction between the two factors ($\eta^2$ = .04). There was also an interaction between sample size and autocorrelation ($\eta^2$ = .02). Inspection of Table 14 reveals acceptable levels of bias when no serial correlation was present in the data and the series length was 5. The level parameter was overestimated across all other conditions.

In general, the magnitude of the overestimation increased as the series length increased. The trend of overestimation across the sample size condition varied as a function of the serial correlation condition, with no clear trend emerging across all conditions (see Table 14). The MARB was greatest under the AR(1) condition where $\varphi$ = .8 when the series length was 8.

After the inadmissible solutions were removed from the data, the simple bias of the estimates of the level mean depended on sample size ($\eta^2$ = .01), series length ($\eta^2$ = .04) and autocorrelation specification ($\eta^2$ = .62). There was also an interaction between series length and autocorrelation ($\eta^2$ = .04). The simple bias remained within acceptable levels across all conditions in the absence of serial correlation, and the level mean was also overestimated across all conditions when serial correlation was present. When serial correlation was present in the data, the simple bias ranged from 0.075 under the ARMA(1, 1) condition where $\varphi$ = .8 and $\theta$ = .3, the sample size was 100, and the series length was 5, to 0.369 under the AR(1) condition where $\varphi$ = .8, the sample size was 1,000, and the series length was 8. In general, the magnitude of the overestimation increased as sample size and series length increased (see Table 14).

Table 14.

*Mean Simple Bias of the Level Parameter under the AR(1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| Including Inadmissible Solutions | | | | | | |
| 5 | 100 | 0.012 | **0.196** | **0.288** | **0.155** | **0.367** |
| | 200 | 0.010 | **0.175** | **0.332** | **0.128** | **0.363** |
| | 500 | -0.001 | **0.156** | **0.369** | **0.120** | **0.366** |
| | 1000 | -0.001 | **0.155** | **0.380** | **0.111** | **0.364** |
| 8 | 100 | **0.173** | **0.287** | **0.389** | **0.281** | **0.372** |
| | 200 | **0.121** | **0.249** | **0.388** | **0.246** | **0.364** |
| | 500 | **0.085** | **0.225** | **0.386** | **0.231** | **0.364** |
| | 1000 | **0.077** | **0.223** | **0.388** | **0.235** | **0.363** |
| Excluding Inadmissible Solutions | | | | | | |
| 5 | 100 | -0.025 | **0.100** | **0.229** | **0.075** | **0.354** |
| | 200 | -0.010 | **0.106** | **0.276** | **0.086** | **0.358** |
| | 500 | -0.005 | **0.125** | **0.321** | **0.112** | **0.366** |
| | 1000 | -0.002 | **0.140** | **0.338** | **0.110** | **0.364** |
| 8 | 100 | 0.017 | **0.141** | **0.337** | **0.191** | **0.341** |
| | 200 | 0.004 | **0.154** | **0.351** | **0.207** | **0.343** |
| | 500 | 0.006 | **0.152** | **0.360** | **0.203** | **0.353** |
| | 1000 | 0.003 | **0.154** | **0.369** | **0.204** | **0.353** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Bias in the estimation of the shape parameter depended on sample size ($\eta^2 = .06$), series length ($\eta^2 = .48$) and autocorrelation specification ($\eta^2 = .91$), as well as an interaction among the three conditions ($\eta^2 = .04$) when inadmissible solutions were included in the data set. As presented in Table 15, bias was within acceptable limits when the series length was 5 and no autocorrelation was present in the data. The model underestimated the shape mean parameter across all other conditions. When serial correlation was present in the data, the relative bias ranged from -0.226 under the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$, the sample size was 1,000, and the series length was 5, to -0.787 under the AR(1) condition where $\varphi = .8$, the sample size was 100, and the series length was 8. In general, the MARB increased as series length increased, but the bias trend across sample sizes varied under the different serial correlation specifications (see Table 15).

After removing inadmissible solutions from the data, bias in the estimation of the shape mean also depended on sample size ($\eta^2 = .08$), series length ($\eta^2 = .23$), autocorrelation specification ($\eta^2 = .93$), and an interaction among the three ($\eta^2 = .02$). The relative bias was within acceptable levels across all conditions when no serial correlation was present in the data set, but the shape mean was underestimated across all conditions when serial correlation was present. Note that a comparison of Tables 14 and 15 reveals that the underestimation of the shape mean mirrors the overestimation of the level mean. As a result, the trends of MARB with regard to the level and shape means are the same across conditions, although the direction of the bias differs.

Table 15.

*Mean Relative Bias of the Shape Parameter under the AR(1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N),*

| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
|---|---|---|---|---|---|---|
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | -0.026 | **-0.388** | **-0.579** | **-0.304** | **-0.737** |
| | 200 | -0.009 | **-0.356** | **-0.665** | **-0.256** | **-0.728** |
| | 500 | < 0.001 | **-0.314** | **-0.738** | **-0.238** | **-0.727** |
| | 1000 | -0.003 | **-0.309** | **-0.758** | **-0.226** | **-0.725** |
| 8 | 100 | **-0.331** | **-0.578** | **-0.787** | **-0.561** | **-0.745** |
| | 200 | **-0.248** | **-0.502** | **-0.783** | **-0.489** | **-0.732** |
| | 500 | **-0.172** | **-0.451** | **-0.779** | **-0.458** | **-0.728** |
| | 1000 | **-0.165** | **-0.447** | **-0.780** | **-0.465** | **-0.727** |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | 0.050 | **-0.196** | **-0.460** | **-0.147** | **-0.709** |
| | 200 | 0.028 | **-0.222** | **-0.556** | **-0.177** | **-0.718** |
| | 500 | 0.007 | **-0.255** | **-0.644** | **-0.219** | **-0.725** |
| | 1000 | -0.001 | **-0.278** | **-0.676** | **-0.224** | **-0.724** |
| 8 | 100 | 0.000 | **-0.282** | **-0.680** | **-0.390** | **-0.684** |
| | 200 | 0.000 | **-0.303** | **-0.709** | **-0.404** | **-0.692** |
| | 500 | -0.006 | **-0.304** | **-0.729** | **-0.404** | **-0.705** |
| | 1000 | -0.004 | **-0.305** | **-0.742** | **-0.404** | **-0.708** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Bias in the estimation of the covariance of the level and shape depended on series length ($\eta^2$ = .21), serial correlation specification ($\eta^2$ = .46), and a two-way interaction between series length and autocorrelation ($\eta^2$ = .19) before inadmissible solutions were removed from the data. There were also two-way interactions between sample size and series length ($\eta^2$ = .01) and sample size and autocorrelation ($\eta^2$ = .03). The MARB decreased as sample size increased across most conditions, although there were exceptions such as the ARMA(1, 1) condition where $\varphi$ = .8 and $\theta$ = .3 which demonstrated the opposite trend (see Table 16). Overall, covariance estimates under 20 of the 40 conditions were within acceptable limits of bias. Of the biased estimates, 8 covariances were overestimated and 12 were underestimated. The overestimated covariances occurred when the series length was large under conditions of little or no serial correlation. In contrast, the underestimated covariances were more likely to occur under ARMA(1, 1) conditions when the series length was small (see Table 16).

After the inadmissible solutions were removed from the data, bias in the estimation of the covariance between the level and the shape depended on sample size ($\eta^2$ = .12), series length ($\eta^2$ = .20), and autocorrelation ($\eta^2$ = .38). There were also two-way interactions between sample size and series length ($\eta^2$ = .07), and between series length and autocorrelation ($\eta^2$ = .22). Inspection of Table 16 reveals that the covariance estimates under 23 of the 40 conditions were within acceptable limits of relative bias after the inadmissible cases were removed. The MARB was smaller under most conditions when compared with the data set that contained inadmissible solutions; however, the MARB increased under the ARMA(1, 1) condition where $\varphi$ = .8 and $\theta$ = .3. In general, estimation improved as the sample size and series length increased, and the magnitude of autocorrelation decreased (see Table 16).

Table 16.

*Mean Simple Bias of the Covariance between the Level and Shape under the Curve of Factors Model with AR(1) parameter by Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
|---|---|---|---|---|---|---|
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | 0.025 | 0.038 | **-0.077** | **-0.061** | **-0.146** |
| | 200 | 0.013 | 0.042 | -0.039 | **-0.076** | **-0.139** |
| | 500 | 0.003 | 0.027 | -0.017 | **-0.076** | **-0.141** |
| | 1000 | 0.001 | 0.024 | -0.008 | **-0.080** | **-0.143** |
| 8 | 100 | **0.120** | **0.092** | 0.011 | -0.040 | 0.024 |
| | 200 | **0.096** | **0.076** | 0.013 | **-0.096** | 0.019 |
| | 500 | **0.077** | **0.064** | 0.011 | **-0.117** | 0.015 |
| | 1000 | **0.078** | **0.064** | 0.011 | **-0.115** | 0.015 |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | **-0.089** | **-0.138** | **-0.135** | **-0.187** | **-0.206** |
| | 200 | -0.047 | **-0.078** | **-0.079** | **-0.134** | **-0.163** |
| | 500 | -0.012 | -0.033 | -0.042 | **-0.092** | **-0.144** |
| | 1000 | -0.003 | -0.006 | -0.026 | **-0.081** | **-0.143** |
| 8 | 100 | 0.005 | 0.010 | -0.028 | **-0.172** | -0.016 |
| | 200 | 0.006 | 0.017 | -0.016 | **-0.162** | -0.006 |
| | 500 | 0.005 | 0.024 | -0.007 | **-0.160** | 0.001 |
| | 1000 | 0.005 | 0.024 | -0.003 | **-0.161** | 0.005 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

ANOVA results indicated that bias in the estimates of the variance of the level depended on sample size ($\eta^2 = .01$), series length ($\eta^2 = .41$), autocorrelation ($\eta^2 = .63$), and an interaction among the three ($\eta^2 = .02$) before the inadmissible solutions were removed from the data. Only 5 of the 40 conditions resulted in variance estimates that were within acceptable limits of bias (see Table 17). As shown in Table 17, the variance of the level was underestimated under most conditions, and the MARB increased as the sample size decreased and series length increased. An exception to this general trend occurred under the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$, whereupon the variance of the level was overestimated and the MARB decreased as series length increased. The mean relative bias of the estimates ranged from -1.477 when no serial correlation was present, the sample size was 100, and the series length was 8, to 1.144 when the sample size was 1,000, the series length was 5, and the ARMA(1, 1) process where $\varphi = .8$ and $\theta = .3$ was present.

After the inadmissible solutions were removed from the data, bias in the estimates of the variance of the level depended on sample size ($\eta^2 = .18$), series length ($\eta^2 = .27$), and autocorrelation ($\eta^2 = .64$), with a three-way interaction among the factors ($\eta^2 = .02$). As when the inadmissible solutions were included in the data set, in general the level variance tended to be underestimated except for under the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$. Note, however, that the variance of the level was underestimated under fewer conditions and overestimated across more conditions. In general, the variance was overestimated when the sample size and series length were small. The variance point estimate decreased as sample size and series length increased such that the variance was underestimated when the sample size and series length were large across most conditions after the inadmissible solutions were removed from the data set (see Table 17).

Closer inspection of the estimated parameters of the model revealed that a contributing factor to the magnitude and direction of the bias was the accuracy of the estimation of the serial correlation process. In general, the level variance was underestimated when the serial correlation process was estimated accurately, and the magnitude of underestimation increased if the serial correlation was overestimated. By contrast, the level variance was overestimated when the serial correlation process was underestimated. The effects of inaccurate estimation of the serial correlation process will be discussed in more detail later in the paper.

Table 17.

*Mean Relative Bias of the Variance of the Level under the AR(1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | **-0.163** | **-0.651** | **0.384** | **0.974** | -0.012 |
| | 200 | **-0.073** | **-0.649** | **-0.105** | **1.109** | **-0.052** |
| | 500 | -0.017 | **-0.534** | **-0.426** | **1.108** | -0.050 |
| | 1000 | -0.010 | **-0.523** | **-0.541** | **1.144** | -0.030 |
| 8 | 100 | **-1.477** | **-1.456** | **-0.652** | -0.001 | **-0.962** |
| | 200 | **-1.193** | **-1.222** | **-0.673** | **0.504** | **-0.907** |
| | 500 | **-0.930** | **-1.063** | **-0.659** | **0.733** | **-0.875** |
| | 1000 | **-0.942** | **-1.052** | **-0.664** | **0.689** | **-0.878** |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | **0.655** | **0.697** | **0.984** | **2.153** | **0.381** |
| | 200 | **0.345** | **0.256** | **0.324** | **1.667** | **0.101** |
| | 500 | **0.083** | **-0.080** | **-0.141** | **1.265** | -0.025 |
| | 1000 | 0.017 | **-0.290** | **-0.328** | **1.160** | -0.027 |
| 8 | 100 | **-0.072** | **-0.359** | **-0.064** | **1.186** | **-0.504** |
| | 200 | **-0.064** | **-0.462** | **-0.254** | **1.111** | **-0.614** |
| | 500 | **-0.066** | **-0.523** | **-0.401** | **1.114** | **-0.712** |
| | 1000 | **-0.062** | **-0.532** | **-0.468** | **1.101** | **-0.764** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Bias in the estimation of the variance of the shape factor also depended on sample size ($\eta^2 = .01$), series length ($\eta^2 = .02$), autocorrelation ($\eta^2 = .78$), and a three-way interaction among them ($\eta^2 = .01$) before inadmissible solutions were removed from the data. The general trends in the estimates of the variance of the shape were similar the trends in the estimates of the variance of the level (see Table 18). The variance of the shape was underestimated under most conditions and the magnitude of the underestimation increased as series length increased, with the exception of the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$ (see Table 18). The mean relative bias ranged from -0.939 under the ARMA(1, 1) condition where $\varphi = .5$ and $\theta = -.3$, the sample size was 100, and the series length was 8, to 2.266 under the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$, the sample size was 500, and the series length was 8.

Analysis of the data set that did not contain inadmissible solutions revealed that bias in the estimation of the variance of the shape was affected by sample size ($\eta^2 = .10$), series length ($\eta^2 = .06$), autocorrelation ($\eta^2 = .83$), and two-way interactions between sample size and series length ($\eta^2 = .05$), and series length and autocorrelation ($\eta^2 = .75$). As presented in Table 18, 31 of the 40 conditions resulted in estimates of the variance of the shape that were considered to be biased. The relative bias tended to decrease in value as sample size and series length increased, at times transitioning from overestimation to underestimation of the parameter. The two exceptions to this general trend were the zero-autocorrelation condition, under which estimation improved as sample size and series length increased, and the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$, under which the parameter value was increasingly overestimated as sample size and series length increased.

Table 18.

*Mean Relative Bias of the Variance of the Shape under the AR(1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | **-0.125** | **-0.640** | **-0.359** | **-0.152** | **-0.225** |
| | 200 | **-0.083** | **-0.634** | **-0.607** | **-0.051** | **-0.256** |
| | 500 | -0.019 | **-0.548** | **-0.759** | -0.035 | **-0.266** |
| | 1000 | -0.015 | **-0.531** | **-0.808** | -0.020 | **-0.262** |
| 8 | 100 | **-0.647** | **-0.918** | **-0.918** | **1.562** | **-0.939** |
| | 200 | **-0.504** | **-0.813** | **-0.927** | **2.044** | **-0.924** |
| | 500 | **-0.381** | **-0.748** | **-0.925** | **2.266** | **-0.913** |
| | 1000 | **-0.377** | **-0.743** | **-0.927** | **2.242** | **-0.912** |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | **0.419** | **0.249** | -0.032 | **0.526** | 0.011 |
| | 200 | **0.208** | -0.027 | **-0.364** | **0.264** | **-0.168** |
| | 500 | **0.053** | **-0.247** | **-0.590** | **0.054** | **-0.252** |
| | 1000 | 0.003 | **-0.379** | **-0.684** | -0.012 | **-0.260** |
| 8 | 100 | -0.002 | **-0.454** | **-0.741** | **2.853** | **-0.809** |
| | 200 | -0.001 | **-0.497** | **-0.803** | **2.730** | **-0.839** |
| | 500 | -0.022 | **-0.519** | **-0.847** | **2.700** | **-0.867** |
| | 1000 | -0.017 | **-0.522** | **-0.868** | **2.708** | **-0.878** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Relative bias in estimates of the standard error of the level depended on sample size ($\eta^2 = .33$), series length ($\eta^2 = .95$), and autocorrelation ($\eta^2 = .85$) before inadmissible solutions were removed from the data. There was also a 3-way interaction among the factors ($\eta^2 = .17$). Only 3 of the 20 conditions resulted in bias outside acceptable limits when series length was 5; however, 19 of the 20 conditions resulted in negative bias outside acceptable limits when series length was 8 (see Table 19). The magnitude of the bias increased as sample size increased when the series length was large. The underestimation of the standard errors was slightly more severe under conditions of small (i.e., AR(1) where $\varphi = .3$) or zero serial correlation (see Table 19).

After the inadmissible solutions were removed from the data, bias in the estimate of the standard error of the level depended on sample size ($\eta^2 = .04$), series length ($\eta^2 = .79$), and autocorrelation ($\eta^2 = .54$), as well as on an interaction among the three factors ($\eta^2 = .47$). Less bias was evident in the standard error estimates after the inadmissible solutions were removed. When series length was 5, 4 of the 20 conditions resulted in biased estimates, and when series length was 8, 8 of the 20 conditions resulted in biased estimates. When the series length was 5, the biased standard error estimates were overestimated, but when the series length was 8, the biased estimates were underestimated. The bias trend across the different samples sizes was inconsistent (see Table 19). Estimation was worst under the AR(1) condition where $\varphi = .8$, the series length was 8, and the sample size was 1,000.

Table 19.

*Mean Relative Bias of the Standard Error of the Level for AR(1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| Including Inadmissible Solutions | | | | | | |
| 5 | 100 | -0.070 | **-0.101** | -0.059 | -0.099 | 0.031 |
| | 200 | -0.046 | **-0.144** | 0.002 | **-0.107** | 0.028 |
| | 500 | -0.025 | -0.077 | 0.097 | -0.032 | 0.078 |
| | 1000 | 0.047 | -0.016 | 0.094 | 0.006 | 0.024 |
| 8 | 100 | **-0.534** | **-0.434** | **-0.173** | **-0.217** | -0.055 |
| | 200 | **-0.603** | **-0.511** | **-0.261** | **-0.212** | **-0.136** |
| | 500 | **-0.679** | **-0.636** | **-0.417** | **-0.292** | **-0.170** |
| | 1000 | **-0.757** | **-0.723** | **-0.484** | **-0.452** | **-0.319** |
| Excluding Inadmissible Solutions | | | | | | |
| 5 | 100 | 0.017 | 0.084 | -0.038 | 0.065 | 0.029 |
| | 200 | -0.006 | **0.134** | 0.035 | 0.069 | 0.024 |
| | 500 | -0.012 | 0.097 | **0.137** | 0.057 | 0.086 |
| | 1000 | 0.059 | **0.103** | **0.166** | 0.020 | 0.027 |
| 8 | 100 | **-0.162** | **-0.147** | **-0.180** | -0.058 | -0.017 |
| | 200 | **-0.137** | -0.058 | **-0.242** | -0.025 | -0.048 |
| | 500 | -0.065 | -0.050 | **-0.358** | 0.000 | -0.058 |
| | 1000 | -0.064 | -0.023 | **-0.435** | -0.030 | **-0.121** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Bias in estimates of the standard error of the shape depended on sample size ($\eta^2$ = .02), series length ($\eta^2$ = .97), autocorrelation ($\eta^2$ = .90), and an interaction among the three factors ($\eta^2$ = .05) when the inadmissible solutions were included in the data set. Estimation of the standard error of the shape was similar to estimation of the standard error of the level in that there was less evidence of bias when the series length was 5, the MARB increased as sample size increased when sample size was 8, and underestimation of the standard errors was slightly more severe under conditions of small or zero serial correlation (see Table 20).

When the series length was 5, 11 of 20 conditions resulted in standard error estimates that were outside acceptable bias limits, but when the series length was 8, all of the standard error estimates were considered to be biased. The biased standard errors were underestimated across most conditions with one exception: under the ARMA(1, 1) condition where $\varphi$ = .5 and $\theta$ = -.3. The standard error estimates tended to be overestimated when the series length was 5. The MARB was greatest when the sample size was 1,000, the series length was 8, and no autocorrelation was present in the data before the inadmissible solutions were removed (see Table 20).

After inadmissible solutions were removed, bias in the standard error of the shape factor continued to be affected by sample size ($\eta^2$ = .23), series length ($\eta^2$ = .89), and autocorrelation ($\eta^2$ = .87) as well as an interaction between the three factors ($\eta^2$ = .60). Overall, 24 of the 40 conditions resulted in estimates that were considered to be biased, with 13 of the biased standard errors occurring when the series length was 5. The direction of the bias tended to be positive when the series length was 5 and negative when the series length was 8. The MARB was greatest under the condition of AR(1) where $\varphi$ = .8, the sample size was 1,000 and the series length was 8 (see Table 20).

Table 20.

*Mean Relative Bias of the Standard Error of the Shape for AR(1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| Including Inadmissible Solutions | | | | | | |
| 5 | 100 | **-0.158** | **-0.229** | **-0.164** | **-0.260** | 0.076 |
| | 200 | -0.060 | **-0.262** | -0.040 | **-0.212** | **0.101** |
| | 500 | -0.022 | **-0.150** | 0.042 | **-0.123** | **0.150** |
| | 1000 | 0.018 | -0.036 | 0.069 | 0.013 | **0.136** |
| 8 | 100 | **-0.802** | **-0.712** | **-0.405** | **-0.473** | **-0.160** |
| | 200 | **-0.829** | **-0.753** | **-0.444** | **-0.475** | **-0.241** |
| | 500 | **-0.862** | **-0.820** | **-0.593** | **-0.553** | **-0.322** |
| | 1000 | **-0.896** | **-0.866** | **-0.681** | **-0.677** | **-0.472** |
| Excluding Inadmissible Solutions | | | | | | |
| 5 | 100 | **0.133** | **0.258** | **-0.126** | **0.198** | 0.099 |
| | 200 | 0.071 | **0.402** | 0.009 | **0.203** | **0.116** |
| | 500 | 0.043 | **0.282** | 0.092 | **0.124** | **0.164** |
| | 1000 | 0.050 | **0.241** | **0.141** | 0.045 | **0.143** |
| 8 | 100 | **-0.187** | -0.074 | **-0.324** | **-0.131** | **0.126** |
| | 200 | **-0.111** | -0.002 | **-0.367** | -0.071 | **0.162** |
| | 500 | **-0.103** | -0.009 | **-0.527** | -0.058 | 0.055 |
| | 1000 | **-0.151** | -0.016 | **-0.629** | -0.080 | -0.063 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

*Fit Criteria*

The proportion of generated data sets for which the chi-square statistic indicated that the AR(1) curve-of-factors model fit the data acceptably is presented in Table 21. The GOF proportions ranged from 0.114 to 0.944. In general, the statistic indicated GOF at a higher rate when the sample size was large and the series length was 5. The GOF proportion was lowest under the ARMA(1, 1) condition where $\varphi = .5$ and $\theta = -.3$ (i.e., when the model was misspecified), when the sample size was 1,000 and the series length was 8.

Table 21.

*Proportion of Generated Data Sets that the Curve-of-Factors Model with AR(1) Parameter was indicated as fitting acceptably by the Chi-Square Statistic while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) ($\varphi = .3$) | AR(1) ($\varphi = .8$) | ARMA(1, 1) ($\varphi = .8, \theta = .3$) | ARMA(1, 1) ($\varphi = .5, \theta = -.3$) |
| 5 | 100 | 0.797 | 0.803 | 0.804 | 0.811 | 0.773 |
| | 200 | 0.888 | 0.898 | 0.904 | 0.902 | 0.816 |
| | 500 | 0.928 | 0.944 | 0.925 | 0.917 | 0.804 |
| | 1000 | 0.947 | 0.948 | 0.944 | 0.930 | 0.638 |
| 8 | 100 | 0.315 | 0.325 | 0.271 | 0.323 | 0.209 |
| | 200 | 0.720 | 0.733 | 0.590 | 0.675 | 0.500 |
| | 500 | 0.855 | 0.892 | 0.716 | 0.763 | 0.414 |
| | 1000 | 0.875 | 0.898 | 0.700 | 0.640 | 0.114 |

*Note*. Estimates based on 1,000 simulations.

The GOF proportions produced by the TLI and RMSEA are presented in Tables 22 and 23. Results from the CFI indicated that the AR(1) model fit each generated data set across each condition (i.e., the GOF was 1.000 across all conditions); therefore no table is presented. For the TLI and RMSEA, the lowest GOF proportion was .873. The GOF proportions were 1.00 across all conditions when sample size was at least 500 (see Tables 22-23). Similar to the results under the curve-of-factors model, the AR(1) model was evaluated as fitting the data well under conditions that resulted in biased parameters.

Table 22.

*Proportion of Generated Data Sets that the Curve-of-Factors Model with AR(1) Parameter was indicated as fitting acceptably by the Tucker-Lewis Index while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | 100 | 1.000 | 1.000 | 1.000 | 0.998 | 0.999 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

*Note*. Estimates based on 1,000 simulations.

Table 23.

*Proportion of Generated Data Sets that the Curve-of-Factors Model with AR(1) Parameter was indicated as fitting acceptably by the Root Mean Squared Error of Approximation while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) ($\varphi$ = .3) | AR(1) ($\varphi$ = .8) | ARMA(1, 1) ($\varphi$ = .8, $\theta$ = .3) | ARMA(1, 1) ($\varphi$ = .5, $\theta$ = -.3) |
| 5 | 100 | 0.931 | 0.930 | 0.945 | 0.925 | 0.911 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | 100 | 0.939 | 0.946 | 0.878 | 0.930 | 0.873 |
| | 200 | 1.000 | 1.000 | 0.999 | 1.000 | 0.999 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

*Note*. Estimates based on 1,000 simulations.

Before inadmissible solutions were removed from the data, ANOVA results indicated that bias in the chi-square statistic depended on sample size ($\eta^2$ = .04), series length ($\eta^2$ = .18), and autocorrelation ($\eta^2$ = .07), as well as an interaction among the three ($\eta^2$ = .01). Inspection of Table 24 reveals that when series length was 5, 8 of 20 conditions resulted in chi-square statistics that were outside acceptable limits of relative bias, but all of the chi-square statistics were outside acceptable bias limits when the series length was 8. When the series length was 5, the MARB decreased as sample size increased, except for the ARMA(1, 1) condition where $\varphi$ = .5 and $\theta$ = -.3 which showed the opposite trend. In general, when the series length was 8 the magnitude of the bias increased as sample size increased.

After the inadmissible solutions were removed from the data, the relative bias of the chi-square statistic depended on sample size ($\eta^2$ = .05), series length ($\eta^2$ = .06), autocorrelation ($\eta^2$ = .07), and a three-way interaction among the factors. When the series length was 5, 8 of the 20 conditions resulted in chi-square statistics that were outside acceptable limits of bias, whereas 16 of the 20 conditions resulted in statistics that were outside acceptable limits of bias when the series length was 8 (see Table 24). In general, the bias decreased as sample size increased; however, under the ARMA(1, 1) condition where $\varphi$ = .5 and $\theta$ = -.3, the chi-square bias was most extreme when the series length was 8 and the sample size was 1,000.

Table 24.

*Mean Relative Bias of the Chi-square Statistic under the Curve of Factors Model with AR(1) parameter by Autocorrelation Specification, Series Length (L) and Sample Size (N) Including Inadmissible Solutions*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | **0.080** | **0.084** | **0.080** | **0.081** | **0.093** |
| | 200 | 0.036 | 0.034 | 0.039 | 0.036 | **0.068** |
| | 500 | 0.017 | 0.012 | 0.012 | 0.017 | **0.080** |
| | 1000 | 0.011 | 0.007 | 0.000 | 0.011 | **0.134** |
| 8 | 100 | **0.181** | **0.150** | **0.163** | **0.145** | **0.172** |
| | 200 | **0.134** | **0.084** | **0.116** | **0.079** | **0.130** |
| | 500 | **0.168** | **0.070** | **0.132** | **0.069** | **0.177** |
| | 1000 | **0.310** | **0.113** | **0.213** | **0.106** | **0.304** |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | **0.082** | **0.080** | **0.078** | **0.081** | **0.094** |
| | 200 | 0.040 | 0.036 | 0.040 | 0.037 | **0.068** |
| | 500 | 0.015 | 0.012 | 0.013 | 0.017 | **0.082** |
| | 1000 | 0.011 | 0.007 | 0.005 | 0.011 | **0.135** |
| 8 | 100 | **0.140** | **0.138** | **0.154** | **0.139** | **0.165** |
| | 200 | **0.067** | **0.064** | **0.100** | **0.074** | **0.111** |
| | 500 | 0.034 | 0.025 | **0.096** | **0.058** | **0.144** |
| | 1000 | 0.029 | 0.020 | **0.134** | **0.083** | **0.243** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

### *Curve-of-factors with ARMA(1, 1) Parameters*

#### *Convergence and Proportion of Inadmissible Solutions*

The convergence rate for the curve-of-factors model across all conditions was 100%. Similar to the AR(1) model, analyzing the data with the ARMA(1, 1) curve-of-factors model resulted in inadmissible solutions across every condition, and increasing the sample size resulted in decreasing proportions of inadmissible solutions (see Table 27). In contrast with the AR(1) model, increasing the series length led to decreasing proportions of inadmissible solutions across most conditions. The proportions of inadmissible solutions ranged from 0.001 when no serial correlation was present in the data, the sample size was 1,000, and the series length was 8 to 0.415 under the AR(1) condition where $\theta = .8$, the sample size was 100, and the series length was 8.

Table 25.

*Proportion of Inadmissible Solutions for the ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) ($\varphi$ = .3) | AR(1) ($\varphi$ = .8) | ARMA(1, 1) ($\varphi$ = .8, $\theta$ = .3) | ARMA(1, 1) ($\varphi$ = .5, $\theta$ = -.3) |
| 5 | 100 | 0.303 | 0.333 | 0.251 | 0.262 | 0.263 |
| | 200 | 0.258 | 0.349 | 0.257 | 0.245 | 0.177 |
| | 500 | 0.157 | 0.344 | 0.324 | 0.194 | 0.044 |
| | 1000 | 0.065 | 0.301 | 0.271 | 0.118 | 0.012 |
| 8 | 100 | 0.095 | 0.275 | 0.415 | 0.155 | 0.296 |
| | 200 | 0.039 | 0.212 | 0.391 | 0.124 | 0.200 |
| | 500 | 0.007 | 0.084 | 0.261 | 0.060 | 0.087 |
| | 1000 | 0.001 | 0.019 | 0.207 | 0.024 | 0.020 |

*Note*. Proportions based on 1,000 simulations.

*Relative and Simple Bias*

The simple bias of the level mean under the curve-of-factors model with ARMA(1, 1) parameters depended series length ($\eta^2$ = .07), autocorrelation specification ($\eta^2$ = .58), and an interaction between the two factors ($\eta^2$ = .13) when the data set contained inadmissible solutions. The simple bias was within acceptable limits under the zero autocorrelation condition; however, the intercept was overestimated across all other conditions (see Table 26). The magnitude of overestimation increased as series length increased and was greatest under the AR(1) condition where $\varphi$ = .8 and the series length was large.

The results were similar after inadmissible solutions were removed from the data, although the magnitude of the bias in the presence of autocorrelation decreased slightly across conditions. ANOVA results indicated that the simple bias of the estimates depended on sample size ($\eta^2 = .03$), series length ($\eta^2 = .13$) and autocorrelation specification ($\eta^2 = .59$), as well as an interaction between series length and autocorrelation ($\eta^2 = .16$). The MARB was greatest when the sample size and series length were large under the two autocorrelation conditions where $\varphi = .8$ (see Table 26).

Table 26.

*Mean Simple Bias of the Level Parameter for the ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | -0.001 | **0.131** | **0.282** | **0.170** | **0.255** |
| | 200 | 0.035 | **0.164** | **0.317** | **0.195** | **0.249** |
| | 500 | 0.026 | **0.179** | **0.360** | **0.182** | **0.249** |
| | 1000 | 0.019 | **0.181** | **0.373** | **0.170** | **0.246** |
| 8 | 100 | 0.005 | **0.174** | **0.386** | **0.360** | **0.263** |
| | 200 | 0.019 | **0.177** | **0.402** | **0.385** | **0.253** |
| | 500 | 0.008 | **0.167** | **0.401** | **0.401** | **0.258** |
| | 1000 | 0.002 | **0.155** | **0.400** | **0.395** | **0.253** |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | **-0.050** | 0.048 | **0.230** | **0.102** | **0.249** |
| | 200 | -0.021 | **0.086** | **0.276** | **0.127** | **0.243** |
| | 500 | -0.014 | **0.103** | **0.311** | **0.131** | **0.247** |
| | 1000 | -0.001 | **0.117** | **0.336** | **0.138** | **0.246** |
| 8 | 100 | -0.024 | **0.098** | **0.350** | **0.333** | **0.248** |
| | 200 | -0.003 | **0.129** | **0.375** | **0.368** | **0.251** |
| | 500 | 0.003 | **0.145** | **0.388** | **0.395** | **0.257** |
| | 1000 | 0.002 | **0.150** | **0.394** | **0.393** | **0.254** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Bias in the estimation of the shape parameter when inadmissible solutions were included in the data set depended on sample size ($\eta^2$ = .04), series length ($\eta^2$ = .36) and autocorrelation specification ($\eta^2$ = .91), as well as an interaction among the three factors ($\eta^2$ = .05). Estimation of the shape parameter was best when no autocorrelation was present in the data (see Table 27). Otherwise, the model underestimated the shape mean across all conditions. In general, the MARB increased slightly as the sample size and series length increased; although there were exceptions to this trend depending on the serial correlation condition (see Table 27). The MARB was greatest when the sample size and series length were large under the two autocorrelation conditions where $\varphi$ = .8.

Bias in the estimation of the shape parameter when inadmissible solutions were removed from the data set also depended on sample size ($\eta^2$ = .17), series length ($\eta^2$ = .55) and autocorrelation specification ($\eta^2$ = .92), as well as an interaction among all three factors ($\eta^2$ = .02). Removal of the inadmissible solutions from the data did not alter the bias trends in the shape mean estimates dramatically (see Table 27). Note that with and without inadmissible solutions in the data the underestimation of the shape parameter was proportional to the overestimation of the level parameter. Overestimation of the level mean corresponded with underestimation of the shape mean. Closer inspection of the data revealed that the $\varphi$ parameter estimate under both the AR(1) and ARMA(1, 1) processes influenced the MARB of the level and shape parameters. This finding will be discussed in more detail later in the paper.

Table 27.

*Mean Relative Bias of the Shape Parameter for the ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) ($\varphi = .3$) | AR(1) ($\varphi = .8$) | ARMA(1, 1) ($\varphi = .8, \theta = .3$) | ARMA(1, 1) ($\varphi = .5, \theta = -.3$) |
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | 0.009 | **-0.255** | **-0.561** | **-0.342** | **-0.510** |
| | 200 | **-0.066** | **-0.332** | **-0.636** | **-0.378** | **-0.499** |
| | 500 | **-0.055** | **-0.354** | **-0.720** | **-0.367** | **-0.500** |
| | 1000 | -0.035 | **-0.361** | **-0.743** | **-0.338** | **-0.495** |
| 8 | 100 | -0.020 | **-0.348** | **-0.778** | **-0.726** | **-0.513** |
| | 200 | -0.040 | **-0.348** | **-0.796** | **-0.768** | **-0.512** |
| | 500 | -0.010 | **-0.334** | **-0.800** | **-0.798** | **-0.511** |
| | 1000 | -0.005 | **-0.311** | **-0.802** | **-0.787** | **-0.507** |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | **0.094** | **-0.100** | **-0.461** | **-0.200** | **-0.490** |
| | 200 | 0.043 | **-0.168** | **-0.550** | **-0.238** | **-0.487** |
| | 500 | 0.022 | **-0.201** | **-0.619** | **-0.265** | **-0.496** |
| | 1000 | 0.004 | **-0.232** | **-0.673** | **-0.274** | **-0.494** |
| 8 | 100 | 0.037 | **-0.197** | **-0.708** | **-0.679** | **-0.489** |
| | 200 | 0.004 | **-0.251** | **-0.747** | **-0.736** | **-0.502** |
| | 500 | -0.002 | **-0.290** | **-0.777** | **-0.785** | **-0.511** |
| | 1000 | -0.004 | **-0.301** | **-0.790** | **-0.783** | **-0.507** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Before inadmissible solutions were removed from the data bias in the estimation of the covariance of the level and shape depended on sample size ($\eta^2 = .04$), series length ($\eta^2 = .13$), autocorrelation specification ($\eta^2 = .12$), and a three-way interaction among them ($\eta^2 = .01$). Results are presented in Table 28. The covariance was slightly underestimated under the two ARMA(1, 1) conditions when series length was 5; otherwise the parameter was estimated well. In general, estimation improved as sample size and series length increased.

Bias in the estimation of the covariance of the level and the shape depended on sample size ($\eta^2 = .25$), series length ($\eta^2 = .40$), and autocorrelation ($\eta^2 = .12$), as well as a three-way interaction among them ($\eta^2 = .02$) after the inadmissible solutions were removed from the data. Removal of the inadmissible cases resulted in 6 more conditions under which the bias was outside acceptable limits (see Table 28). In general, the bias trend was similar to the trend observed before the inadmissible solutions were removed. Estimation of the covariance of the level and shape was worst under the ARMA(1, 1) conditions when series length was 5.

Table 28.

*Mean Simple Bias of the Covariance of the Level and Shape under the ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| Including Inadmissible Solutions | | | | | | |
| 5 | 100 | -0.044 | **-0.056** | **-0.078** | **-0.098** | **-0.106** |
| | 200 | -0.029 | -0.021 | -0.041 | **-0.072** | **-0.111** |
| | 500 | -0.004 | 0.007 | -0.012 | **-0.055** | **-0.114** |
| | 1000 | -0.008 | 0.020 | -0.008 | **-0.059** | **-0.111** |
| 8 | 100 | -0.020 | 0.011 | 0.011 | **-0.083** | 0.011 |
| | 200 | -0.009 | 0.016 | 0.015 | -0.048 | 0.011 |
| | 500 | -0.002 | 0.020 | 0.014 | -0.020 | 0.012 |
| | 1000 | -0.002 | 0.021 | 0.016 | -0.020 | 0.011 |
| Excluding Inadmissible Solutions | | | | | | |
| 5 | 100 | **-0.177** | **-0.223** | **-0.140** | **-0.211** | **-0.214** |
| | 200 | **-0.108** | **-0.136** | **-0.084** | **-0.151** | **-0.158** |
| | 500 | -0.034 | **-0.065** | -0.040 | **-0.099** | **-0.124** |
| | 1000 | -0.016 | -0.026 | -0.026 | **-0.083** | **-0.113** |
| 8 | 100 | -0.026 | -0.011 | -0.012 | **-0.117** | -0.016 |
| | 200 | -0.011 | 0.005 | 0.001 | **-0.068** | -0.001 |
| | 500 | -0.003 | 0.016 | 0.008 | -0.026 | 0.009 |
| | 1000 | -0.002 | 0.020 | 0.013 | -0.022 | 0.011 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Estimates of the variance of the level were affected by sample size ($\eta^2$ = .05), series length ($\eta^2$ = .21), autocorrelation ($\eta^2$ = .31) and an interaction among the three before the inadmissible solutions were removed from the data ($\eta^2$ = .02). The mean relative bias ranged from -0.830 to 1.106 as presented in Table 29. The variance of the level was estimated best when no serial correlation was present in the data. In general, the point estimate of variance of the level decreased as sample size and series length increased, which is reflected by mean relative bias estimates that transition from positive bias to negative bias across most autocorrelation conditions.

Bias in the estimates of the variance of the level also depended on sample size ($\eta^2$ = .31), series length ($\eta^2$ = .48), autocorrelation ($\eta^2$ = .44), and a three-way interaction among the factors ($\eta^2$ = .01) after inadmissible solutions were removed from the data. The mean relative bias ranged from -0.776 to 2.169 (see Table 29). The general trend of the bias across conditions was similar to the trend observed before inadmissible solutions were removed, although more variances were overestimated and the magnitude of overestimation was greater after the inadmissible solutions were removed. The variance tended to be overestimated when sample size and series length were small but decreased steadily such that the variance tended to be underestimated when series length and sample size were large. Two exceptions to this trend occurred under the zero autocorrelation and ARMA(1, 1) $\varphi$ = .8 and $\theta$ = .3 conditions, whereupon the variance estimates seemed to stabilize as sample size and series length increased (see Table 29).

Table 29.

*Mean Relative Bias of the Variance of the Level for ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| Including Inadmissible Solutions | | | | | | |
| 5 | 100 | **0.152** | 0.011 | **0.403** | **1.106** | **0.060** |
| | 200 | **0.078** | **-0.268** | -0.041 | **0.835** | **0.110** |
| | 500 | -0.026 | **-0.452** | **-0.423** | **0.708** | **0.110** |
| | 1000 | 0.012 | **-0.536** | **-0.515** | **0.772** | **0.095** |
| 8 | 100 | **0.062** | **-0.450** | **-0.681** | **0.444** | **-0.637** |
| | 200 | 0.025 | **-0.488** | **-0.781** | **0.180** | **-0.642** |
| | 500 | 0.001 | **-0.519** | **-0.804** | -0.048 | **-0.645** |
| | 1000 | 0.012 | **-0.500** | **-0.830** | -0.028 | **-0.630** |
| Excluding Inadmissible Solutions | | | | | | |
| 5 | 100 | **1.112** | **1.231** | **0.985** | **2.169** | **0.795** |
| | 200 | **0.679** | **0.618** | **0.374** | **1.637** | **0.428** |
| | 500 | **0.220** | **0.161** | **-0.110** | **1.211** | **0.175** |
| | 1000 | **0.089** | **-0.110** | **-0.315** | **1.070** | **0.111** |
| 8 | 100 | **0.178** | **-0.104** | **-0.319** | **0.766** | **-0.329** |
| | 200 | **0.078** | **-0.293** | **-0.559** | **0.390** | **-0.506** |
| | 500 | 0.011 | **-0.441** | **-0.708** | 0.015 | **-0.612** |
| | 1000 | 0.013 | **-0.480** | **-0.776** | -0.007 | **-0.623** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Bias in the estimation of the variance of the shape factor depended on sample size ($\eta^2 = .09$), series length ($\eta^2 = .05$), autocorrelation ($\eta^2 = .47$), and a three-way interaction among those factors ($\eta^2 = .04$) before inadmissible cases were removed from the data. The relative bias ranged from -.959 to .692 (see Table 30). As with estimation of the variance of the level, the point estimates of the variance of the shape tended to decrease as sample size and series length increased. The two exceptions to this trend occurred under the zero autocorrelation and ARMA(1, 1) $\varphi = .8$ and $\theta = .3$ conditions, under which the estimation of the variance of the shape seemed to stabilize as sample size and series length increased.

Removal of the inadmissible solutions from the data set did not alter the general trend of bias in the estimation of the variance of the shape. The magnitude of relative bias also depended on sample size ($\eta^2 = .28$), series length ($\eta^2 = .26$), autocorrelation ($\eta^2 = .52$), and a three-way interaction among them ($\eta^2 = .04$). The relative bias ranged from -0.944 to 1.007 (see Table 30). The general trend was similar to that observed for estimation of the variance of the level in that more conditions were considered to be positively biased after the inadmissible solutions were removed from the data. In addition, the point estimates of the variance decreased steadily as sample size series length increased, except for estimates under the zero autocorrelation and ARMA(1, 1) $\varphi = .8$ and $\theta = .3$ conditions, which seemed to stabilize as sample size and series length increased (see Table 30).

Table 30.

*Mean Relative Bias of the Variance of the Shape for ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | **0.478** | 0.034 | **-0.321** | 0.010 | **-0.152** |
| | 200 | **0.174** | **-0.255** | **-0.550** | **-0.176** | **-0.151** |
| | 500 | 0.000 | **-0.444** | **-0.754** | **-0.265** | **-0.155** |
| | 1000 | 0.022 | **-0.522** | **-0.793** | **-0.242** | **-0.168** |
| 8 | 100 | **0.172** | **-0.447** | **-0.904** | **0.692** | **-0.708** |
| | 200 | **0.058** | **-0.499** | **-0.938** | **0.293** | **-0.710** |
| | 500 | 0.018 | **-0.527** | **-0.951** | 0.001 | **-0.716** |
| | 1000 | 0.008 | **-0.517** | **-0.959** | 0.014 | **-0.713** |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | **0.995** | **0.871** | 0.013 | **0.614** | **0.256** |
| | 200 | **0.604** | **0.409** | **-0.302** | **0.285** | 0.029 |
| | 500 | **0.227** | **0.021** | **-0.558** | 0.004 | **-0.118** |
| | 1000 | **0.103** | **-0.206** | **-0.669** | **-0.092** | **-0.159** |
| 8 | 100 | **0.250** | **-0.228** | **-0.799** | **1.007** | **-0.623** |
| | 200 | **0.111** | **-0.374** | **-0.874** | **0.494** | **-0.673** |
| | 500 | 0.028 | **-0.475** | **-0.924** | **0.066** | **-0.708** |
| | 1000 | 0.009 | **-0.505** | **-0.944** | 0.035 | **-0.711** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Bias in estimates of the standard error of the level depended on sample size ($\eta^2$ = .33), series length ($\eta^2$ = .54), and autocorrelation ($\eta^2$ = .80), as well as an interaction among the three factors ($\eta^2$ = .41) when inadmissible solutions were included in the data. As presented in Table 31, 6 of the 20 conditions when series length was 5 resulted in standard error bias outside acceptable limits, while 10 of the 20 conditions when series length was 8 resulted in bias outside acceptable limits. The mean relative bias ranged from 0.380 to -0.283. Estimation of the standard error was worst when no serial correlation was present in the data, and estimation was best under the ARMA(1, 1) condition where $\varphi$ = .5 and $\theta$ = -.3 (see Table 31).

After the inadmissible solutions were removed from the data set, bias in the estimate of the standard error of the level also depended on sample size ($\eta^2$ = .64), series length ($\eta^2$ = .81), autocorrelation ($\eta^2$ = .60), and a three-way interaction among the factors ($\eta^2$ = .24). There was more bias evident in the standard error estimates after the inadmissible solutions were removed when series length was 5, but less bias was evident when series length was 8 (see Table 31). The mean relative bias ranged from -0.187 to 0.401. Estimation of the standard error was much improved under the zero-autocorrelation condition after inadmissible cases were removed from the data. The standard error tended to be overestimated under the AR(1) conditions when series length was 5 and sample size was large.

Table 31.

*Mean Relative Bias of the Standard Error of the Level for the ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| | | Including Inadmissible Solutions | | | | |
| 5 | 100 | **-0.193** | -0.064 | 0.028 | -0.034 | 0.030 |
| | 200 | **-0.172** | 0.033 | 0.056 | -0.009 | -0.017 |
| | 500 | **-0.108** | -0.015 | **0.217** | -0.043 | 0.082 |
| | 1000 | **-0.203** | **-0.107** | **0.380** | -0.079 | 0.072 |
| 8 | 100 | **-0.258** | **-0.225** | **-0.116** | **-0.128** | -0.006 |
| | 200 | **-0.283** | **-0.212** | -0.099 | -0.081 | 0.043 |
| | 500 | **-0.157** | **-0.205** | -0.048 | **-0.109** | 0.062 |
| | 1000 | -0.079 | -0.085 | 0.003 | **-0.150** | 0.027 |
| | | Excluding Inadmissible Solutions | | | | |
| 5 | 100 | -0.076 | 0.014 | 0.057 | 0.073 | 0.064 |
| | 200 | -0.052 | **0.149** | 0.055 | **0.199** | -0.012 |
| | 500 | 0.098 | **0.288** | **0.262** | **0.264** | 0.085 |
| | 1000 | 0.027 | **0.334** | **0.401** | **0.323** | 0.078 |
| 8 | 100 | **-0.165** | **-0.111** | **-0.137** | **-0.112** | -0.020 |
| | 200 | **-0.187** | 0.008 | -0.071 | -0.085 | 0.027 |
| | 500 | -0.089 | 0.056 | -0.047 | **-0.101** | 0.050 |
| | 1000 | -0.062 | 0.033 | 0.045 | **-0.141** | 0.023 |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

Before inadmissible solutions were removed from the data, bias in the estimate of the standard error of the shape depended on sample size ($\eta^2 = .60$), series length ($\eta^2 = .62$), autocorrelation ($\eta^2 = .88$), and a three-way interaction among them ($\eta^2 = .41$). As presented in Table 32, when series length was 5, 12 of the 20 conditions resulted in standard error estimates outside acceptable limits of bias. When series length was 8, 17 of 20 conditions resulted in standard error estimates that were considered to be biased. The mean relative bias ranged from -0.389 to 0.367. All of the standard errors estimated under the zero-autocorrelation condition were outside acceptable limits of bias. The MARB tended to decrease as sample size increased, but there were exceptions to this general trend (e.g., ARMA(1, 1) where $\varphi = .5$ and $\theta = -.3$).

Bias in the standard error of the shape factor depended on sample size ($\eta^2 = .80$), series length ($\eta^2 = .84$), and autocorrelation ($\eta^2 = .76$), and an interaction between the three factors ($\eta^2 = .46$) after inadmissible solutions were removed from the data. Estimation improved under the large series length conditions after the inadmissible solutions were removed, with 12 of the 20 conditions resulting in biased standard errors. The mean relative bias ranged from -0.284 to 0.438. The magnitude of standard error underestimation was less under the large series length condition when compared with the results including inadmissible solutions; however, the magnitude of overestimation when series length was small increased (see Table 32).

Table 32.

*Mean Relative Bias of the Standard Error of the Shape under the ARMA(1, 1) Curve-of-factors Model while Varying Autocorrelation Specification, Series Length (L), and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| Including Inadmissible Solutions | | | | | | |
| 5 | 100 | **-0.292** | **-0.128** | -0.026 | **-0.150** | 0.019 |
| | 200 | **-0.249** | 0.004 | -0.003 | -0.092 | 0.074 |
| | 500 | **-0.160** | -0.044 | **0.212** | -0.082 | **0.152** |
| | 1000 | **-0.260** | **-0.140** | **0.367** | **-0.137** | **0.113** |
| 8 | 100 | **-0.389** | **-0.350** | **-0.256** | **-0.243** | -0.011 |
| | 200 | **-0.374** | **-0.305** | **-0.174** | **-0.203** | 0.063 |
| | 500 | **-0.213** | **-0.271** | **-0.111** | **-0.244** | **0.132** |
| | 1000 | **-0.104** | **-0.116** | -0.039 | **-0.253** | **0.200** |
| Excluding Inadmissible Solutions | | | | | | |
| 5 | 100 | **-0.143** | -0.014 | 0.010 | 0.014 | 0.037 |
| | 200 | -0.097 | **0.175** | 0.004 | **0.184** | 0.096 |
| | 500 | 0.071 | **0.310** | **0.271** | **0.336** | **0.178** |
| | 1000 | -0.007 | **0.430** | **0.394** | **0.438** | **0.120** |
| 8 | 100 | **-0.284** | **-0.144** | **-0.236** | **-0.221** | -0.028 |
| | 200 | **-0.255** | 0.046 | **-0.107** | **-0.191** | 0.067 |
| | 500 | **-0.128** | 0.098 | -0.052 | **-0.232** | **0.119** |
| | 1000 | -0.077 | 0.061 | 0.063 | **-0.242** | **0.215** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent parameter estimates that were considered to be biased.

*Fit Criteria*

The proportions of simulated data sets for which the chi-square statistic indicated that the ARMA(1, 1) curve-of-factors model fit the data acceptably across conditions are presented in Table 33. The GOF proportions range from 0.321 to 0.953. In general, the statistic indicated GOF at a slightly higher rate when the sample size was 5 than when the sample size was 8. The GOF proportion also increased as the sample size increased. The lowest proportion occurred when the sample size was 100 and the series length was 8 under the AR(1) condition where $\varphi = .8$ (i.e., when the model was misspecified).

Table 33.

*Proportion of Generated Data Sets that the Curve-of-Factors Model with ARMA(1, 1) Parameters was indicated as fitting acceptably by the Chi-Square Statistic while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) ($\varphi = .3$) | AR(1) ($\varphi = .8$) | ARMA(1, 1) ($\varphi = .8, \theta = .3$) | ARMA(1, 1) ($\varphi = .5, \theta = -.3$) |
| 5 | 100 | 0.811 | 0.798 | 0.803 | 0.794 | 0.792 |
| | 200 | 0.890 | 0.893 | 0.914 | 0.893 | 0.889 |
| | 500 | 0.929 | 0.933 | 0.930 | 0.930 | 0.883 |
| | 1000 | 0.935 | 0.938 | 0.953 | 0.946 | 0.864 |
| 8 | 100 | 0.360 | 0.336 | 0.321 | 0.324 | 0.333 |
| | 200 | 0.745 | 0.755 | 0.754 | 0.750 | 0.725 |
| | 500 | 0.915 | 0.914 | 0.905 | 0.895 | 0.848 |
| | 1000 | 0.949 | 0.936 | 0.924 | 0.925 | 0.793 |

*Note*. Estimates based on 1,000 simulations.

The GOF proportions produced by the TLI and RMSEA are presented in Tables 34 and 35. The GOF proportions produced by the CFI were 1.000 under each condition, with the exception of the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$, the series length was 8, and the sample size was 100 where the GOF was .999. Therefore, no table is presented for the CFI. Similar to the results under the previous two models, the GOF proportions were very high across all conditions. Across all three fit criteria, the lowest GOF proportion was 0.928. The GOF proportion was 1.000 across all conditions when the sample size was at least 200 (see Tables 34 - 35). Thus, as was seen for the two previously examined models, there were conditions under which the ARMA(1, 1) model produced biased parameter estimates (e.g., AR(1) where $\varphi = .8$) yet was evaluated as fitting the data well.

Table 34.

*Proportion of Generated Data Sets that the Curve-of-Factors Model with ARMA(1, 1) Parameters was indicated as fitting acceptably by the Tucker-Lewis Index while Varying Autocorrelation Specification, Series Length (L) and Sample Size (N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| 5 | 100 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | 100 | 1.000 | 1.000 | 1.000 | 0.998 | 1.000 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

*Note*. Estimates based on 1,000 simulations.

Table 35.

*Proportion of Generated Data Sets that the Curve-of-Factors Model with ARMA(1, 1) Parameters was indicated as fitting acceptably by the Root Mean Squared Error of Approximation while Varying Autocorrelation Specification, Series Length (L) and Sample Size(N)*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) ($\varphi = .3$) | AR(1) ($\varphi = .8$) | ARMA(1, 1) ($\varphi = .8, \theta = .3$) | ARMA(1, 1) ($\varphi = .5, \theta = -.3$) |
| 5 | 100 | 0.936 | 0.931 | 0.937 | 0.934 | 0.928 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | 100 | 0.932 | 0.944 | 0.937 | 0.935 | 0.942 |
| | 200 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 500 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 1000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

*Note*. Estimates based on 1,000 simulations.

Before inadmissible solutions were removed from the data, ANOVA results indicated that bias in the chi-square statistic depended on sample size ($\eta^2 = .10$) and series length ($\eta^2 = .01$). Inspection of Table 36 reveals that when series length was 5, 6 of 20 conditions resulted in chi-square statistics that were outside acceptable limits of bias. When the series length was 8, 11 of 20 conditions resulted in chi-square statistics that were considered to be biased. In general, the magnitude of the bias decreased as sample size increased.

After the inadmissible solutions were removed from the data, ANOVA results indicated that the relative bias of the chi-square statistic depended on sample size ($\eta^2$ = .10). As depicted in Table 55, 6 of the 20 conditions resulted in chi-square statistics that were outside acceptable limits of bias when the series length was 5, whereas 11 of the 20 conditions resulted in statistics that were outside acceptable limits of bias when the series length was 8. In general, the bias decreased as sample size increased meaning the model was evaluated as fitting the data better as sample size increased (see Table 36).

Table 36.

*Mean Relative Bias of the Chi-square Statistic under the Curve of Factors Model with ARMA(1, 1) Parameters by Autocorrelation Specification, Series Length (L) and Sample Size (N) Including Inadmissible Solutions*

| | | Data Generating Model | | | | |
|---|---|---|---|---|---|---|
| L | N | Zero Autocorrelation | AR(1) (φ = .3) | AR(1) (φ = .8) | ARMA(1, 1) (φ = .8, θ = .3) | ARMA(1, 1) (φ = .5, θ = -.3) |
| Including Inadmissible Solutions | | | | | | |
| 5 | 100 | **0.076** | **0.076** | **0.078** | **0.076** | **0.085** |
| | 200 | 0.035 | 0.040 | 0.039 | 0.037 | 0.042 |
| | 500 | 0.011 | 0.013 | 0.015 | 0.010 | 0.041 |
| | 1000 | 0.007 | 0.008 | 0.005 | 0.007 | **0.057** |
| 8 | 100 | **0.135** | **0.135** | **0.136** | **0.138** | **0.137** |
| | 200 | **0.063** | **0.062** | **0.060** | **0.058** | **0.069** |
| | 500 | 0.021 | 0.023 | 0.021 | 0.022 | 0.042 |
| | 1000 | 0.009 | 0.008 | 0.011 | 0.018 | **0.051** |
| Excluding Inadmissible Solutions | | | | | | |
| 5 | 100 | **0.077** | **0.080** | **0.079** | **0.078** | **0.082** |
| | 200 | 0.036 | 0.040 | 0.036 | 0.039 | 0.043 |
| | 500 | 0.015 | 0.014 | 0.015 | 0.012 | 0.041 |
| | 1000 | 0.008 | 0.011 | 0.004 | 0.007 | **0.057** |
| 8 | 100 | **0.134** | **0.136** | **0.136** | **0.138** | **0.136** |
| | 200 | **0.063** | **0.060** | **0.061** | **0.058** | **0.068** |
| | 500 | 0.021 | 0.023 | 0.022 | 0.023 | 0.041 |
| | 1000 | 0.009 | 0.008 | 0.012 | 0.016 | **0.051** |

*Note*. Estimates based on 1,000 simulations. Bolded values represent estimates that were considered to be biased.

*Comparison of the Three Models*

I compared the three models with respect to convergence rates and proportions of inadmissible solutions, unbiased estimation of population parameters and standard errors, and production of fit criteria that would support the hypothesized model. Because maximum likelihood estimation can result in inadmissible solutions, all inadmissible solutions had to be removed from the final data sets and additional admissible solutions had to be generated to replace the inadmissible solutions.

The reason for obtaining 1,000 sets of results for each condition with each method is that the models were compared across replications. When comparing the models, the means and standard deviations of the parameter estimates and fit criteria should be obtained from the same number of observations. Therefore, the results in this section are based on analysis of the complete data sets without inadmissible solutions.

*Convergence and Proportion of Inadmissible Solutions*

All of the models converged across all conditions; however, there were differences in the number of inadmissible solutions produced across conditions. The curve-of-factors model performed better than the two models that incorporated serial correlation parameters with respect to the proportion of admissible solutions output across conditions. The proportion of inadmissible solutions generated by the curve-of-factors model was zero across every condition but one; for that one condition, the proportion of inadmissible solutions was less than .01.

By contrast, the proportion of inadmissible solutions produced by the AR(1) and ARMA(1, 1) models was substantial across many conditions. Bearing in mind the superior performance of the curve-of-factors model, this section will describe differences between the AR(1) and ARMA(1, 1) models in the proportion of inadmissible solutions produced across conditions.

Across conditions, both models tended to have fewer inadmissible solutions as sample size increased (see Figure 6). The ARMA(1, 1) model tended to perform better as series length increased, but the AR(1) model did not necessarily follow this trend. Across the two ARMA(1, 1) serial correlation conditions, the AR model performed better under the smaller series length condition. Under the other three conditions the performance of the model under each series length of was influenced by the sample size.

Likewise, whether the ARMA(1, 1) or AR(1) model performed better depended on sample size, series length,  and the magnitude of serial correlation present in the data (see Figure 6). In general, the ARMA(1, 1) model with a sample size of 1,000 and a series length of 8 consistently performed well.

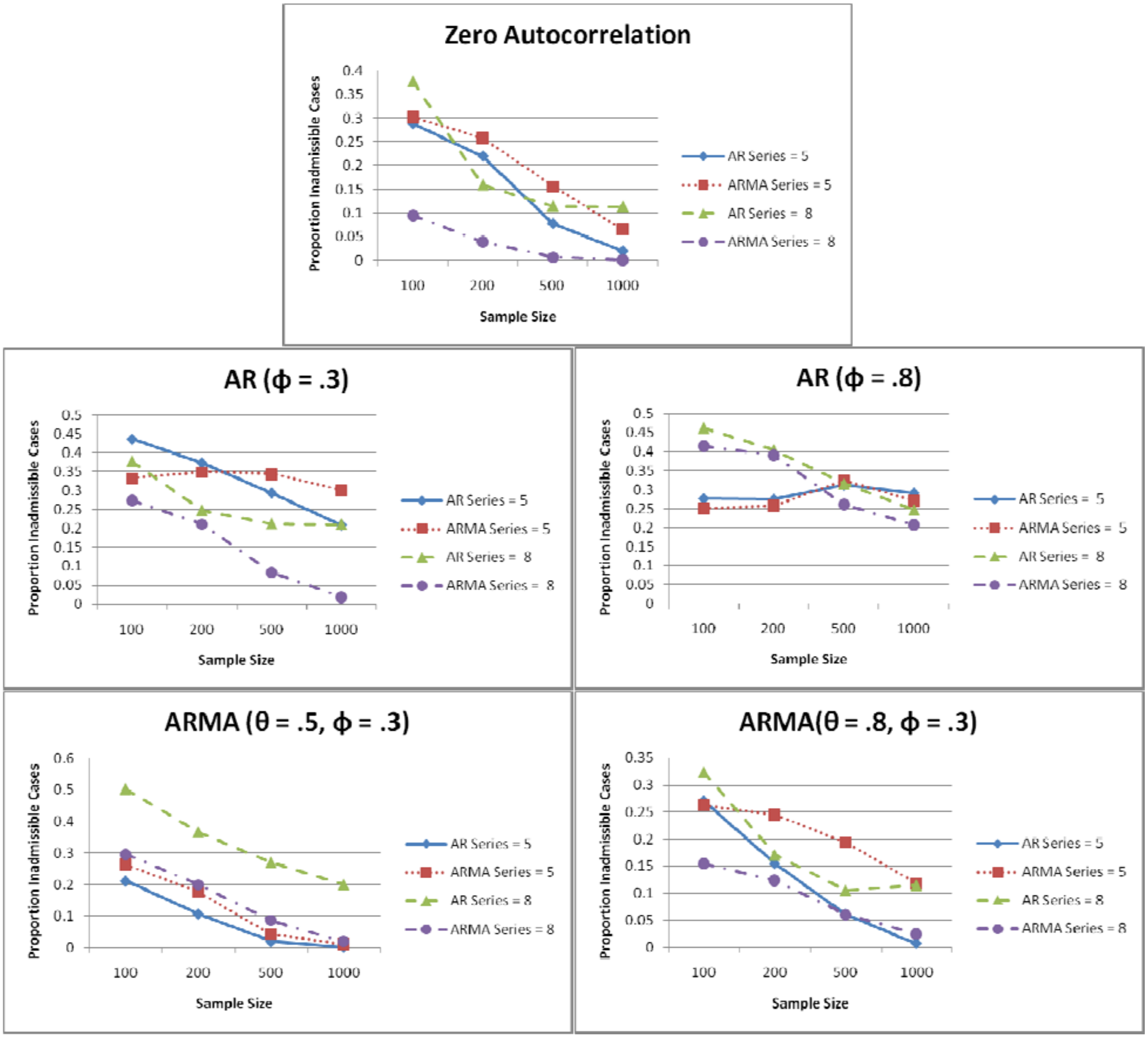

Figure 6. Proportions of inadmissible cases across conditions for the AR(1) and ARMA(1, 1) curve-of-factors models

*Relative and Simple Bias*

I examined differences in relative bias among models by using repeated measures MANOVA, with the model specification treated as the repeated measures factor. Therefore, model specification differences were indicated by a statistically significant difference in the repeated measures factor. MANOVA results indicated that the simple bias of the level mean depended on the model specification ($\eta^2 = .60$), with a three-way interaction among the model specification, series length, and serial correlation magnitude present in the data ($\eta^2 = .06$). The curve-of-factors model performed better than the two models that included serial correlation parameters (see Figure 7). Estimation of the level mean was unbiased under the curve-of-factors model across all conditions, whereas the level mean estimates were positively biased across non-zero serial correlation conditions under the AR(1) and ARMA(1, 1) models.

A comparison of the AR(1) and ARMA(1, 1) models revealed similar levels of positive bias across most conditions, although the ARMA(1, 1) model exhibited less bias under the ARMA(1, 1) condition where $\varphi = .5$ and $\theta = -.3$. The ARMA(1, 1) model produced greater bias in the estimate of the level under the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$ when the series length was 8 (see Figure 7).

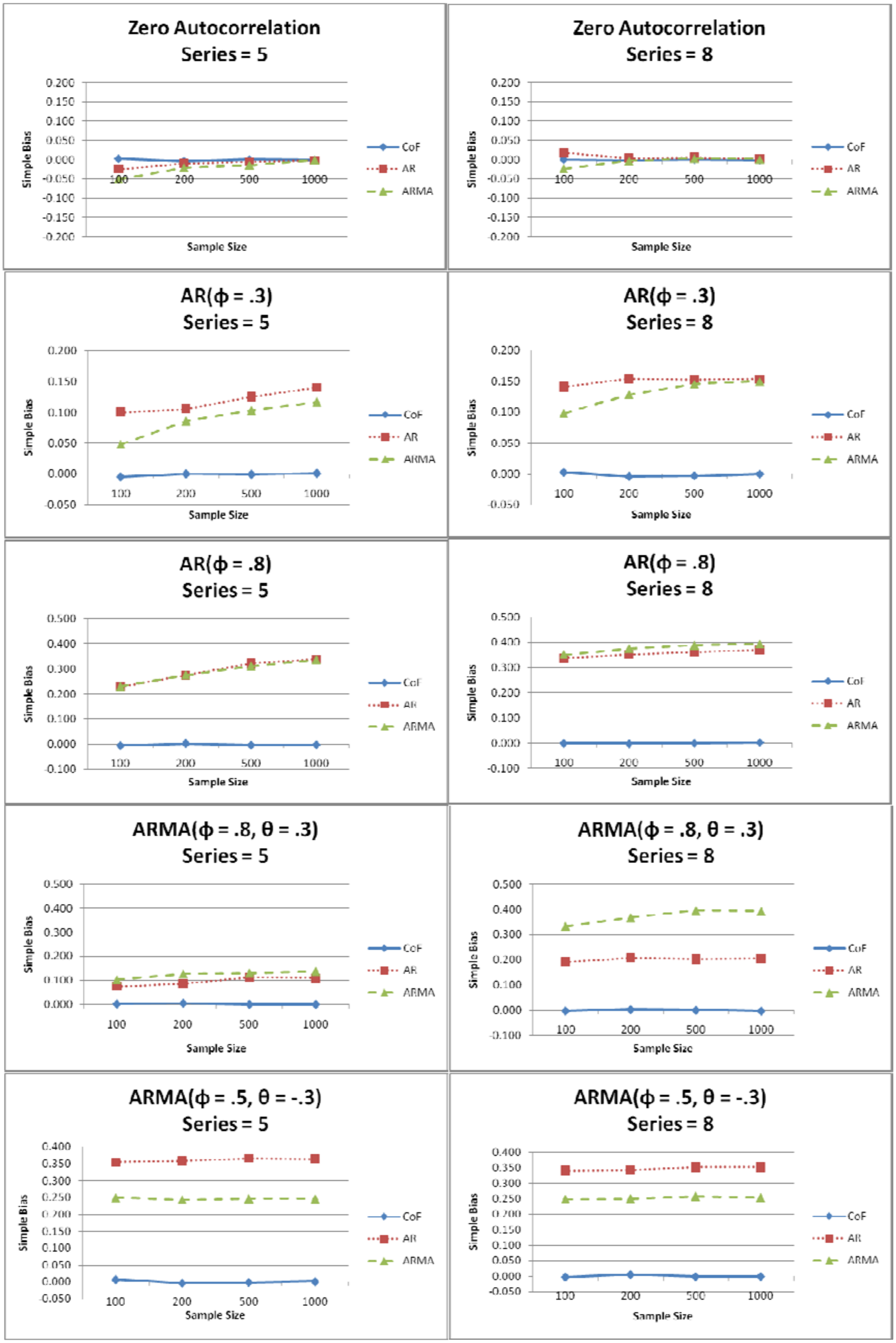


Figure 7. Mean simple bias of estimates of the level mean for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

The magnitude of the bias of the shape mean depended on the estimating model ($\eta^2$ = .78) and a three-way interaction among the estimating model, series length, and magnitude of autocorrelation present in the data ($\eta^2$ = .13). As with the level mean, estimation of the shape mean was unbiased across all conditions under the curve-of-factors model; however, estimation of the shape mean was negatively biased under the AR(1) and ARMA(1, 1) models across all non-zero serial correlation conditions.

A comparison of Figure 7 and Figure 8 reveals that the bias of the shape mirrors the bias of the level. For example, the ARMA(1, 1) model exhibited less bias than the AR(1) model under the ARMA(1, 1) condition where $\varphi$ = .5 and $\theta$ = -.3 for both the level and shape estimates. For both parameter estimates, the trend lines across sample size are approximately parallel. The ARMA model exhibited less positive bias in the estimation of the level, and it exhibited less negative bias in the estimation of the shape. Thus, the bias trends in estimates of the shape are identical to the bias trends in estimates of the level, although the direction of the bias differs.

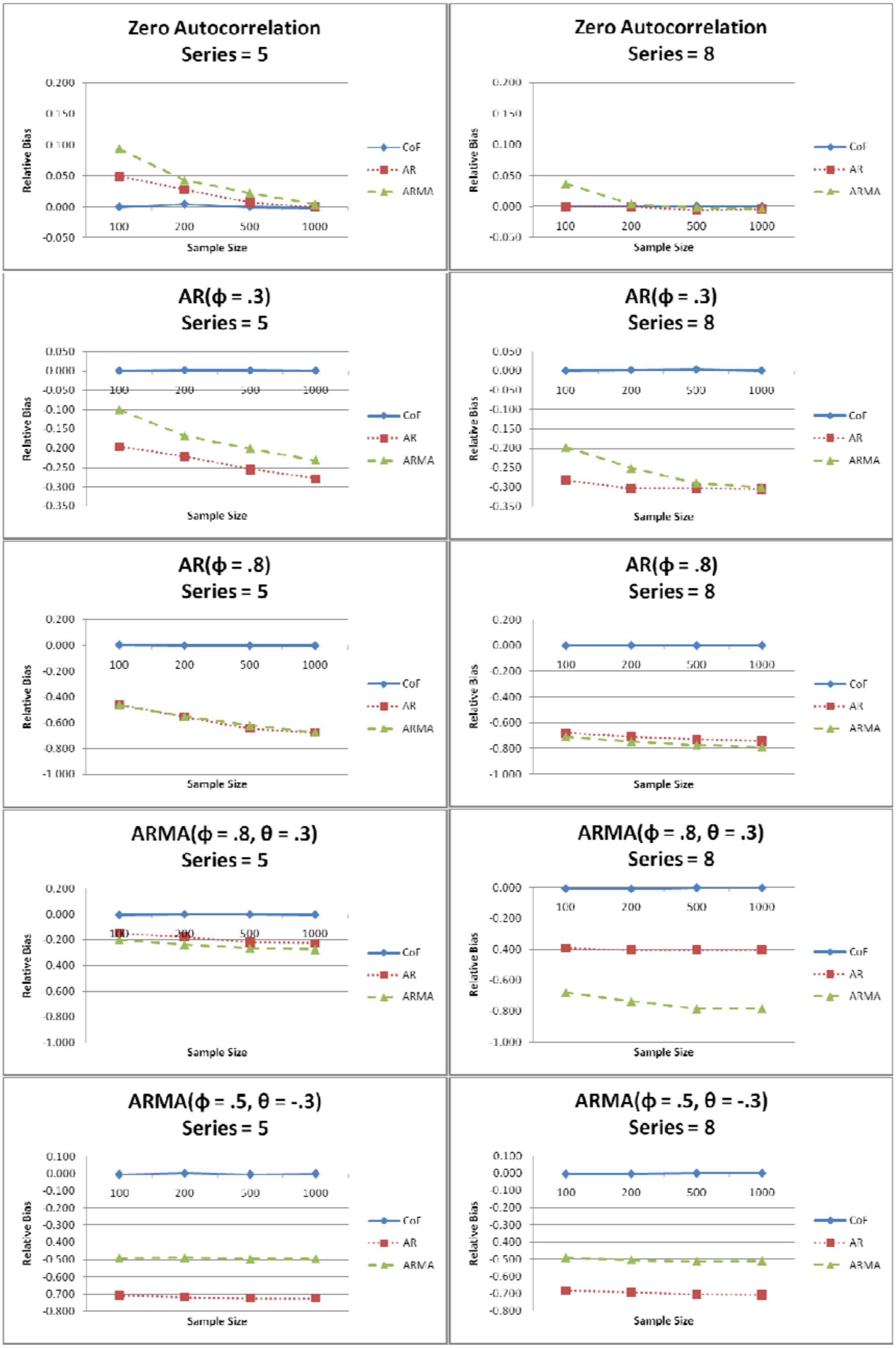


Figure 8. Mean relative bias of estimates of the shape mean for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

Bias in the estimation of the covariance of the level and shape was influenced by the specified model ($\eta^2$ = .48). There were also two three-way interactions: one among the estimating model, series length, and magnitude of autocorrelation in the data ($\eta^2$ = .15) and the other between the estimating model, sample size, and series length ($\eta^2$ = .03). The covariance of the level and shape was estimated accurately under the curve-of-factors model in the absence of autocorrelation; however, estimates were downwardly biased across conditions in the presence of autocorrelation (see Figure 9).

In general, the covariance parameter was estimated more accurately under the AR(1) and ARMA(1, 1) models when serial correlation was present in the data. Under both models, the covariance of the level and shape tended to be underestimated when sample size and series length were small, with estimation improving as sample size and series length increased. Both models also estimated the covariance parameter accurately across most conditions when the series length was 8, with the exception of the ARMA(1, 1) condition where $\varphi$ = .8 and $\theta$ = .3. Under that combination of conditions, the AR(1) model consistently underestimated the parameter. The ARMA(1, 1) model also underestimated the parameter when sample size was small, but estimation improved as sample size increased (see Figure 9).

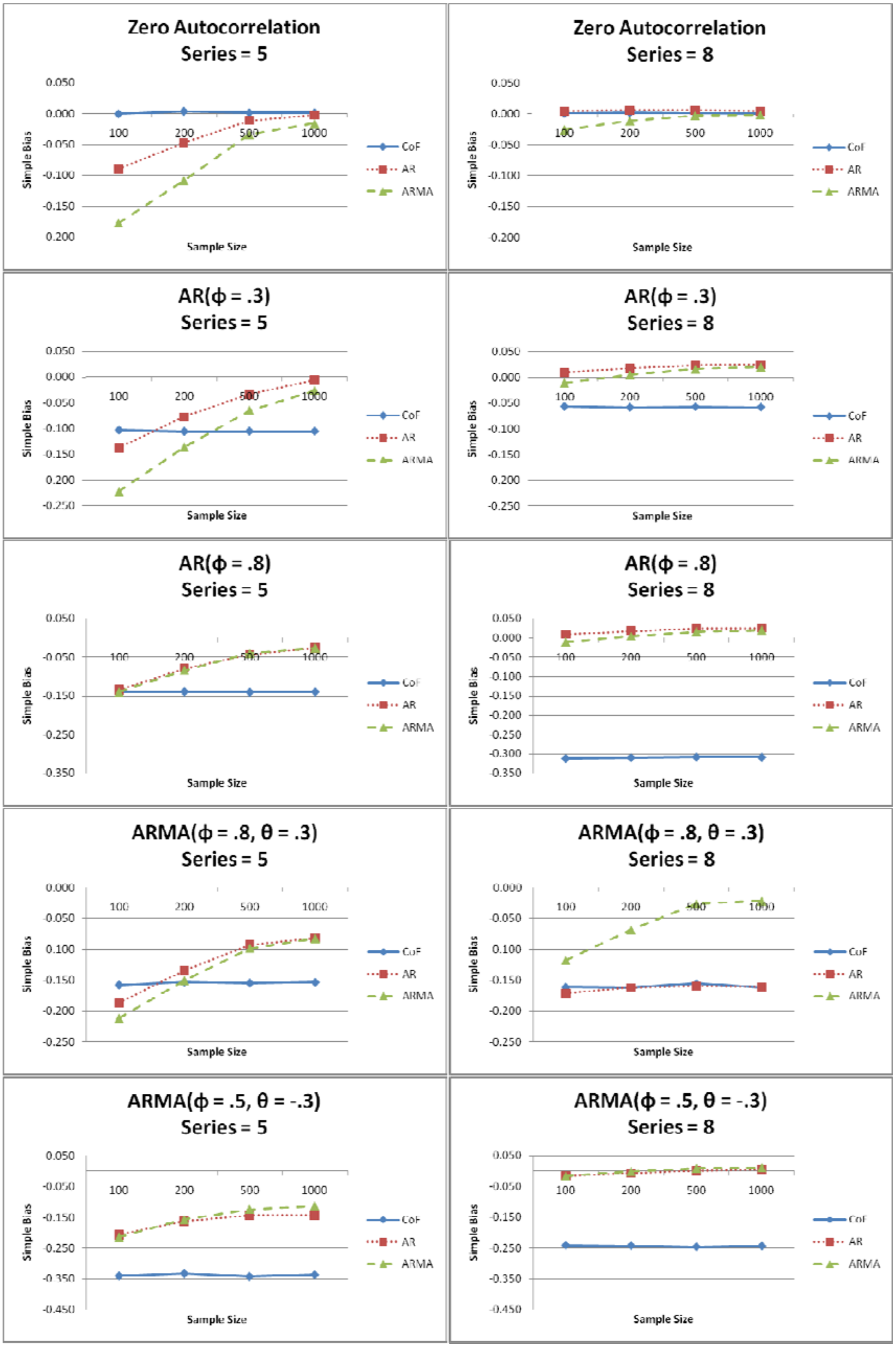


Figure 9. Mean simple bias of estimates of the covariance of the level and shape for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

Relative bias in estimates of the variance of the level depended on the specified model ($\eta^2 = .83$) and two three-way interactions: one interaction among model, sample size, and series length ($\eta^2 = .02$), and the other interaction among model, series length, and autocorrelation magnitude ($\eta^2 = .36$). Estimation of the variance of the level was good under the curve-of-factors model when no autocorrelation was present in the data (see Figure 10). The variance estimates were outside acceptable limits of bias under the AR(1) and ARMA(1, 1) models when the sample size and series length were small, but estimation improved as sample size and series length increased.

Estimates of the variance of the level were poor across all non-zero serial correlation conditions under all models. The curve-of-factors model overestimated the variance across all conditions, with the magnitude of the bias tending to increase as the magnitude of the autocorrelation present in the data increased. Under the AR(1) condition where $\varphi = .8$, the estimated variance was more than 5 times the nominal variance. By contrast, the AR(1) and ARMA(1, 1) models tended to overestimate the variance when sample size and series length were small, but tended to underestimate the variance when sample size and series length were large. The MARB tended to be smaller under the AR(1) and ARMA(1, 1) models than under the curve-of-factors model (see Figure 10).

The AR(1) model performed slightly better than the ARMA(1, 1) model across most conditions, with the exception of the ARMA(1, 1) condition where $\varphi = .8$ and $\theta = .3$ when the series length was 8. Under that condition, the ARMA(1, 1) model performed better than the AR(1) model (see Figure 10).

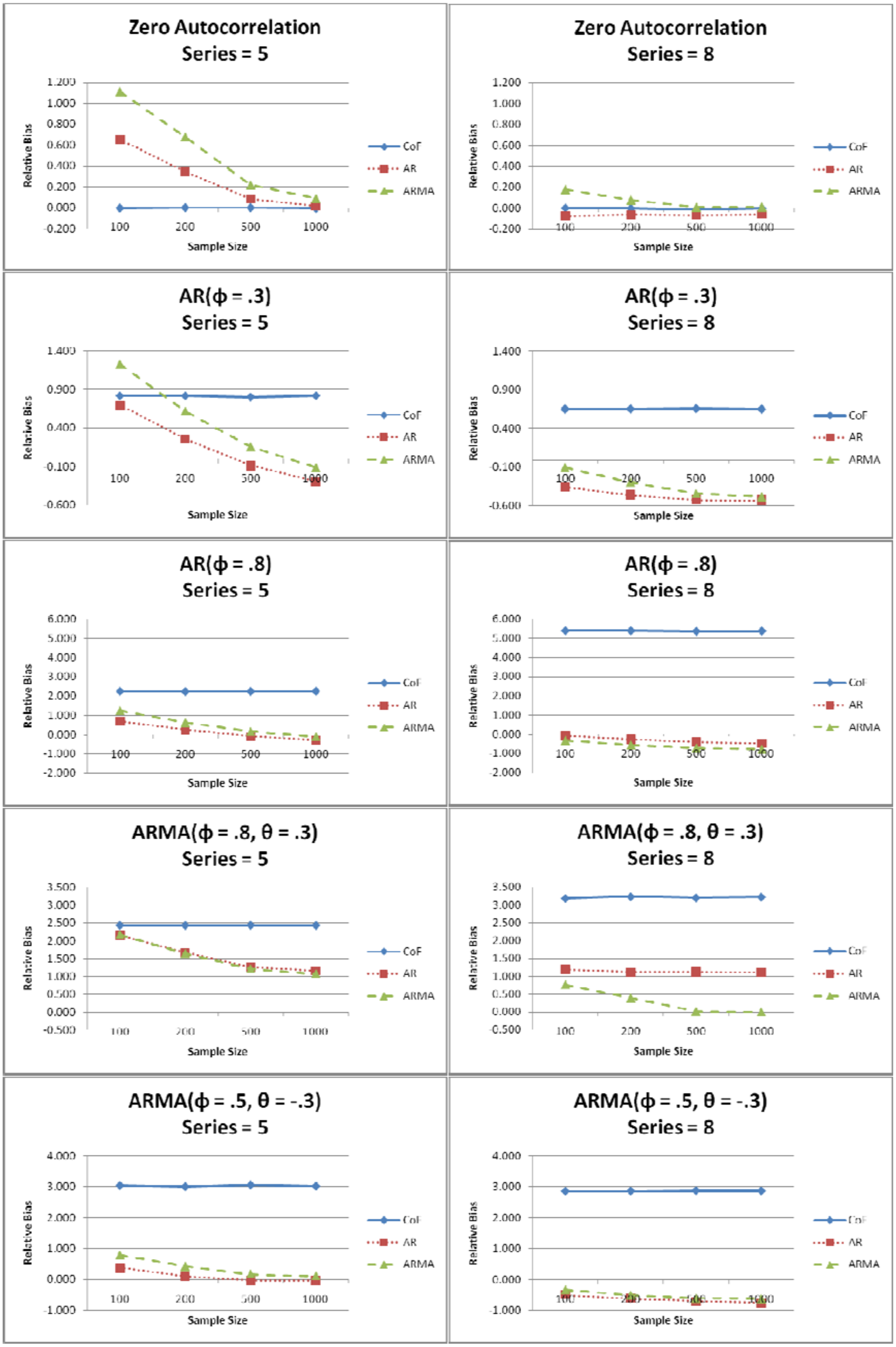


Figure 10. Mean relative bias of estimates of the variance of the level for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

Relative bias in estimates of the variance of the shape depended on the model specification ($\eta^2$ = .78) and a four-way interaction among model specification, sample size, series length, and autocorrelation magnitude ($\eta^2$ = .01). The bias trends in estimation of the variance of the shape were similar to the bias trends in estimation of the variance of the level (see Figure 11). The curve-of-factors model estimated the variance of the shape accurately in the absence of serial correlation but overestimated the variance of the shape when serial correlation was present in the data. The ARMA(1, 1) condition where $\varphi$ = .8 and $\theta$ = .3 was again the condition that produced the greatest magnitude of bias under the curve-of-factors model, whereupon the estimated variance was more than 6 times the nominal variance. In general the MARB was greater under the curve-of-factors model than the AR(1) and ARMA(1, 1) models when serial correlation was present in the data.

The AR(1) and ARMA(1, 1) showed similar trends of bias across most conditions. The AR(1) model performed slightly better when the magnitude of the serial correlation was small, and the ARMA(1, 1) model performed better under the ARMA(1, 1) condition where $\varphi$ = .8 and $\theta$ = .3 when series length was 8 (see Figure 11).

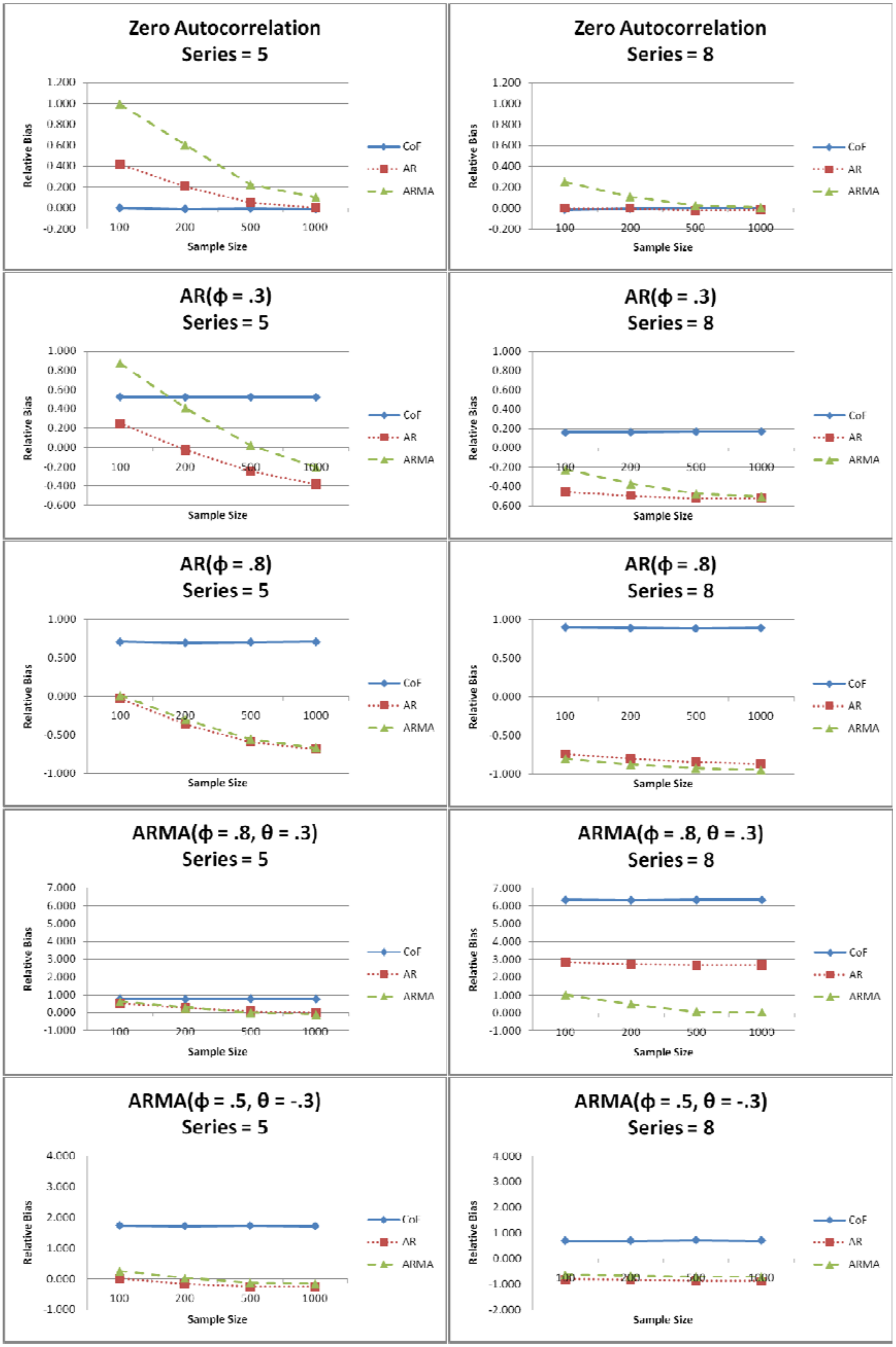


Figure 11. Mean relative bias of estimates of the variance of the shape for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

Relative bias in estimates of the standard error of the level depended on the model specification ($\eta^2$ = .02) and a four-way interaction among model specification, sample size, series length, and autocorrelation magnitude ($\eta^2$ = .01). The curve-of-factors model performed best, with estimates of the standard error of the level within acceptable limits of bias across all conditions (see Figure 12). Under the AR(1) model, standard error estimates were generally within acceptable limits when sample size and series length were small but were more inconsistent when the sample size and series length were large. The AR(1) model performed better than the ARMA(1, 1) model when the series length was small, but results were mixed when the series length was large.

When the series length was large, the AR(1) and ARMA(1, 1) models performed similarly when there was little or no autocorrelation in the data. Under the other conditions, the AR(1) model performed better under the ARMA(1, 1) condition where $\varphi$ = .8 and $\theta$ = .3, but the ARMA(1, 1) model performed better under the ARMA(1, 1) condition where $\varphi$ = .5 and $\theta$ = -.3 and the AR(1) condition where $\varphi$ = .8 (see Figure 12).

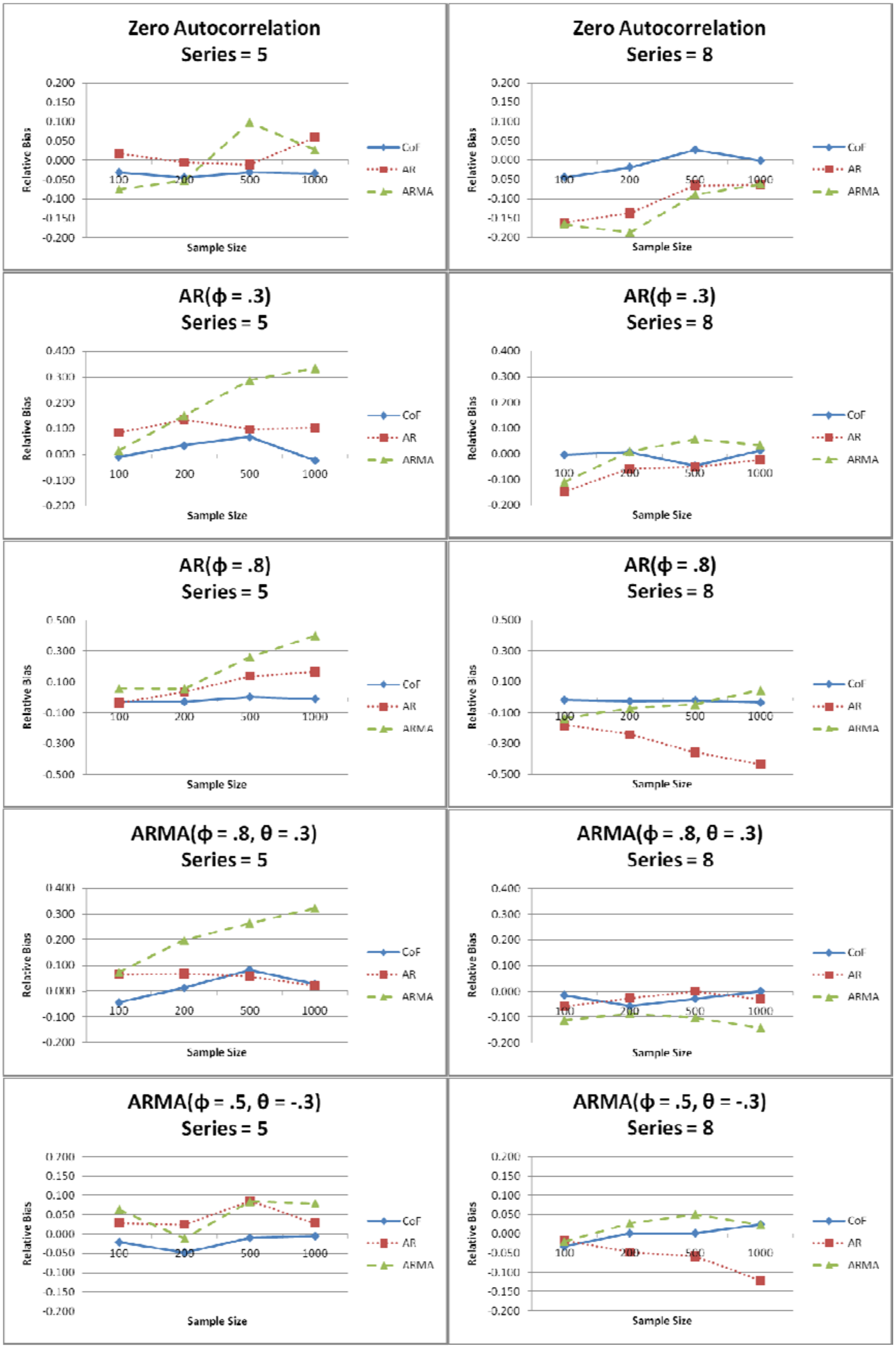


Figure 12. Mean relative bias of estimates of the standard error of the level for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

Relative bias in estimates of the standard error of the shape depended on a four-way interaction among model, sample size, series length, and autocorrelation magnitude ($\eta^2$ = .02). Estimates of the standard error of the shape were considered to be unbiased across all conditions under the curve-of-factors model (see Figure 13). As with estimates of the standard error of the level, the performance of the curve-of-factors model was the best of the three models.

There was an interaction between the AR(1) and ARMA(1, 1) models under several conditions when the series length was small such that the standard error estimates were better under the ARMA(1, 1) model when the sample size was small, but the estimates were better under the AR(1) model when the sample size was large. The trend was less clear when the series length was 8, with each model performing better under specific sample sizes and autocorrelation specifications, but neither model exhibiting superior performance overall (see Figure 13).

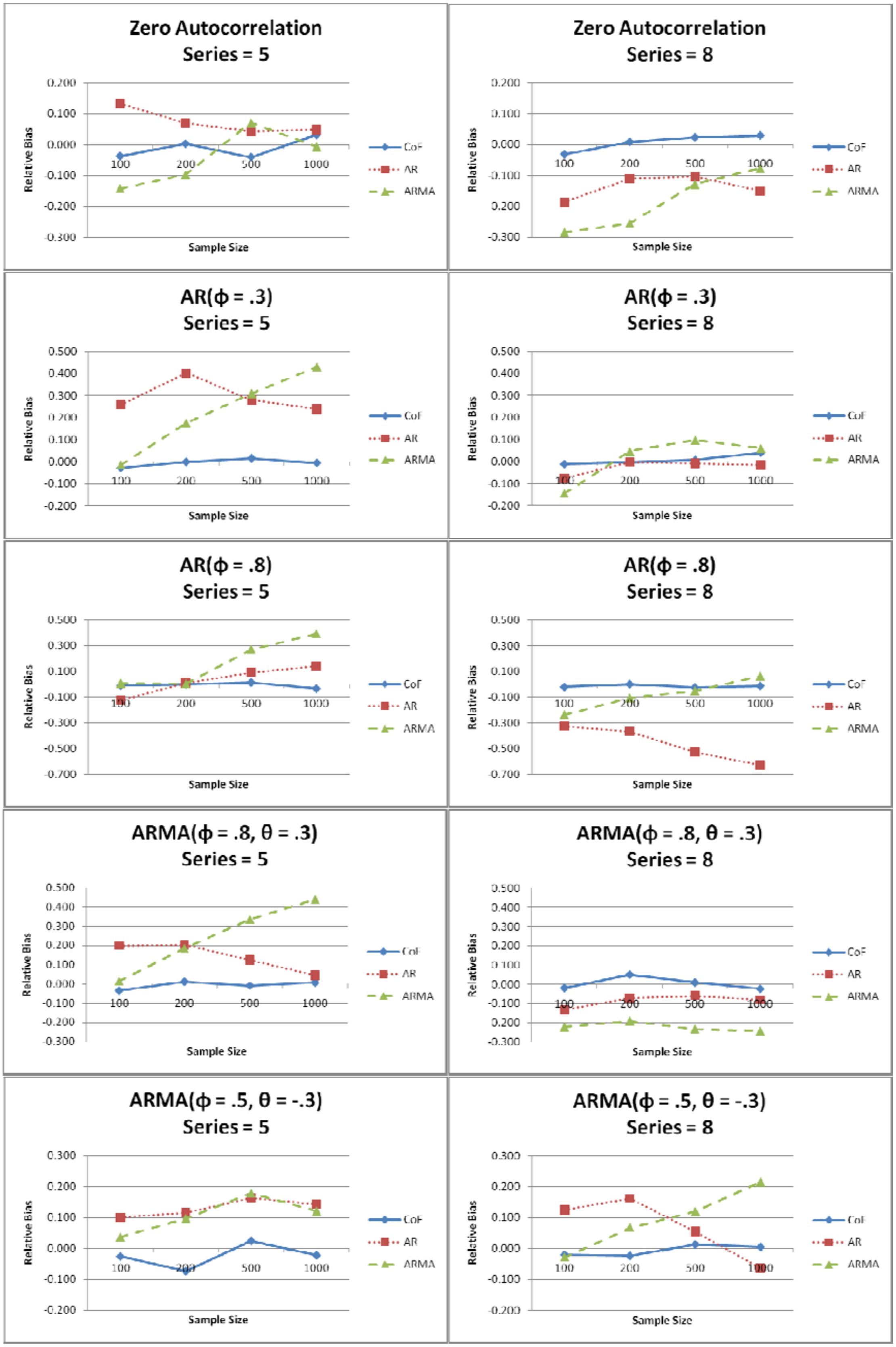


Figure 13. Mean relative bias of estimates of the standard error of the shape for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

*Fit Criteria*

The models were evaluated as to how well they fit the generated data across conditions by the CFI, TLI, RMSEA and chi-square statistics. The models were compared on the GOF proportions produced by the fit criteria across each condition. There was very little difference among the three models in the GOF proportions as indicated by the TLI and CFI; for each model across each condition, the proportion was either equal to or very close to 1.00. In other words, the TLI and CFI almost always indicated that each of the models fit the data well across every condition.

The RMSEA was only slightly more discriminating. The AR(1) and ARMA(1, 1) models were consistently indicated by the RMSEA as fitting the data well across each condition. The curve-of-factors model, in contrast, was indicated as fitting the data well across most but not all conditions by the RMSEA. The conditions under which the curve-of-factors model was not indicated by the RMSEA as adequately fitting the data included the ARMA(1, 1) serial correlation conditions, particularly the condition where $\varphi = .5$ and $\theta = -.3$.

The chi-square statistic, as expected, was more discriminating than the CFI, TLI, and RMSEA. When sample size and series length were small, the GOF proportions for all of the models were high (i.e., > .700), with one exception: the GOF proportion for the curve-of-factors model was .308 under the ARMA condition $\varphi = .5$ and $\theta = -.3$. The GOF proportion as indicated by the chi-square statistic for the curve-of-factors model approached zero as sample size and series length increased under non-zero autocorrelation conditions.

By contrast, the AR(1) and ARMA(1, 1) models were indicated as having adequate fit more consistently across conditions. The GOF proportions as indicated by the chi-square statistic for the two models were similar when series length was small; however, when series length was large, the ARMA(1, 1) model was indicated by the chi-square statistic as fitting the data well more often than the AR(1) model, particularly under the ARMA(1, 1) conditions (i.e., when the ARMA(1, 1) model was correctly specified and the AR(1) model was misspecified).

Relative bias in the chi-square statistic depended on model specification ($\eta^2 = .84$) and a four-way interaction among model specification, sample size, series length, and autocorrelation magnitude ($\eta^2 = .19$). The relative bias was similar for all models in the absence of autocorrelation (see Figure 14), when bias was unacceptable at small sample sizes but within acceptable limits for large sample sizes.

When autocorrelation was present in the data, the chi-square relative bias was outside acceptable limits for the curve-of-factors model across most conditions (see Figure 14), and the magnitude of the bias increased as sample size and series length increased. By contrast, the AR(1) and ARMA(1, 1) models were within acceptable bias limits across non-zero autocorrelation conditions when series length was small. When series length was large, the chi-square bias was outside acceptable limits for the AR(1) model under the large-magnitude autocorrelation conditions. The ARMA(1, 1) had chi-square statistics within acceptable limits across most conditions (see Figure 14).

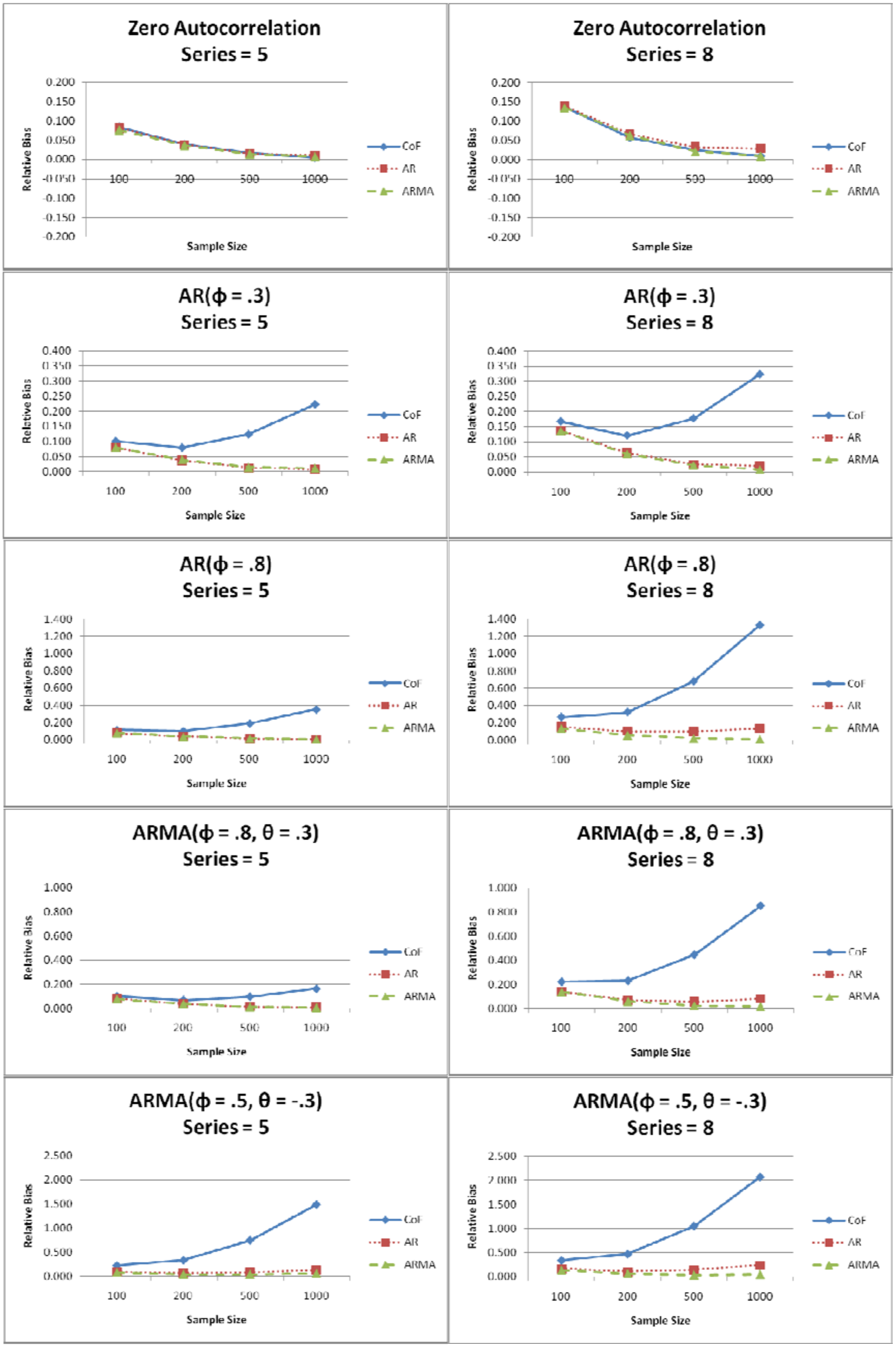


Figure 14. Mean relative bias of the chi-square statistic for the curve-of-factors, AR(1) curve-of-factors, and ARMA(1, 1) curve-of-factors models across study conditions

# Chapter V: Discussion

This study was motivated by two overarching goals: the first was to examine the effects of serial correlation on growth parameter estimates of interest under the curve-of-factors growth model; the second was an attempt to modify the curve-of-factors model to measure growth and autocorrelation processes within the same data set. The study builds upon and was inspired by previous research by Curran and Bollen (2001) and Sivo et al. (2005). The research by Sivo et al. found that unmodeled serial correlation can result in biased growth parameter estimates under a first-order latent growth model, and the research by Curran and Bollen (2001) developed a first-order ALT model designed to measure serial correlation and growth within the same data set.

A search of the literature found neither methodological studies examining the effects of serial correlation on more sophisticated, second-order latent growth models such as the curve-of-factors model, nor second-order latent growth models modified to measure both serial correlation and growth. My study compared the performance of the curve-of-factors model with two similar models that were specified with either an autoregressive parameter or autoregressive and moving-average parameters, under varying sample size, series length, and serial correlation processes. This section of the dissertation will present conclusions that can be drawn from the model comparisons as well as limitations of the study and directions for future research. The first part of this section will present conclusions regarding the effects of serial correlation on growth parameters of interest under the curve-of-factors model. The second part of this section will describe conclusions in terms of the attempt to measure growth and serial correlation processes within the same set of data, and the third part of this section will present the limitations of the study and directions for future research.

The results of the model comparisons have shown that the curve-of-factors model is able to produce unbiased estimates of the fixed effects and their standard errors when autocorrelation is present but unmodeled in the data set. This finding adds to a growing body of research indicating that growth models in general are able to accurately reproduce the fixed effects parameter values when serial correlation is present in the data but ignored by the model (Ferron et al., 2002; Kwok et al., 2007; Murphy & Pituch, 2009; Sivo et al., 2005). However, this study also adds to findings from the same body of research indicating that the variance components are poorly estimated by growth models in the presence of serial correlation.

It appears that failure to model serial correlation using the curve-of-factors model results in overestimation of both the variance of the level and the variance of the shape, and underestimation of their covariance. An analysis of the model's covariance structure can give insight as to the causes of the bias. Equation 29 presents the covariance structure of the first- and second-order factors for the curve-of-factors model. The $\boldsymbol{\Psi}$ matrix in this equation, which is the first-order factor covariance matrix, is the specific covariance matrix that is misspecified when serial correlation is present in the first-order factor structure. In particular, the off diagonal elements of the $\boldsymbol{\Psi}$ matrix are specified to be zero under the curve-of-factors model when the true values of the off diagonal elements are non-zero. Because the model is missing parameters, the true composite covariance matrix cannot be reproduced exactly. However, computation of Equation 29 with a misspecified $\boldsymbol{\Psi}$ matrix and a $\boldsymbol{\Phi}$ matrix that combines overestimated variances and an underestimated covariance can result in a model-implied composite covariance matrix that is reasonably close to the true composite covariance matrix.

Thus, variance overestimation and covariance underestimation will increase under the curve-of-factors model as the magnitude of the serial correlation in the data increases, because the magnitude of the off diagonal elements in the $\mathbf{\Psi}$ matrix increases as serial correlation increases. Because the true structures of the model's covariance matrices rarely are known in practice, fit criteria are usually used to gauge whether or not the discrepancy between the true composite covariance matrix and the model-implied covariance matrix is acceptable. Unfortunately, this study has shown that biased variance components estimates under the curve-of-factors model will not necessarily be reflected in the values produced by fit criteria.

Using Hu and Bentler's (1999) suggested combination of CFI $\geq 0.95$, TLI $\geq 0.95$, and RMSEA $\leq 0.05$ to gauge acceptable model fit, the curve-of-factors model would have been retained as fitting the data well under most conditions evaluated in this study, even under conditions under which the random parameters were estimated poorly. Although there were small sample size conditions under which the model would have been rejected at a high rate (e.g., ARMA (1, 1) where $\varphi = .5$ and $\theta = -.3$), the model would have been evaluated as fitting the data well at least 95% of the time when the sample size was at least 500. Researchers should therefore be cautious when drawing inferences about the variance components under the curve-of-factors model if there is reason to believe that serial correlation is present in the data.

Because previous research has indicated that unmodeled serial correlation can cause growth models to produce biased variance components estimates, the second goal of this dissertation was an attempt to measure both growth and autocorrelation processes within the same data set. The two curve-of-factors models that incorporated serial correlation parameters were able to produce estimates of the growth and serial correlation parameters; unfortunately, the models also produced biased estimates of parameters of interest.

The results of my study indicate that adding serial correlation parameters to the curve-of-factors model may produce biased estimates of the fixed effects, their standard errors, and variance components when serial correlation is present in the data. With respect to the fixed effects estimates, the point estimate of the level parameter was overestimated and the point estimate of the shape parameter was underestimated. Closer inspection of the model implied trajectory revealed, however, that the growth trajectory was not estimated as poorly as the relative bias of the fixed effects estimates would indicate.

Although the estimates of the level and shape means were considered to be biased, when the level and shape estimates were combined with the AR(1) parameter estimate, the result was approximately linear with a slope of 0.5. For example, under the condition where the series length was 8 and ARMA(1, 1) serial correlation parameters were $\varphi = .5$ and $\theta = -.3$, the AR(1) model estimated $\varphi = .70$, the level mean = 0.35, and the shape mean = 0.15. The ARMA(1, 1) model estimated $\varphi = .5$, the level mean as 0.25, and the shape mean as 0.25. Substituting these values into Equation 18 produces trajectories that converge on the true growth trajectory values as the series length increases, as depicted in Figure 15.

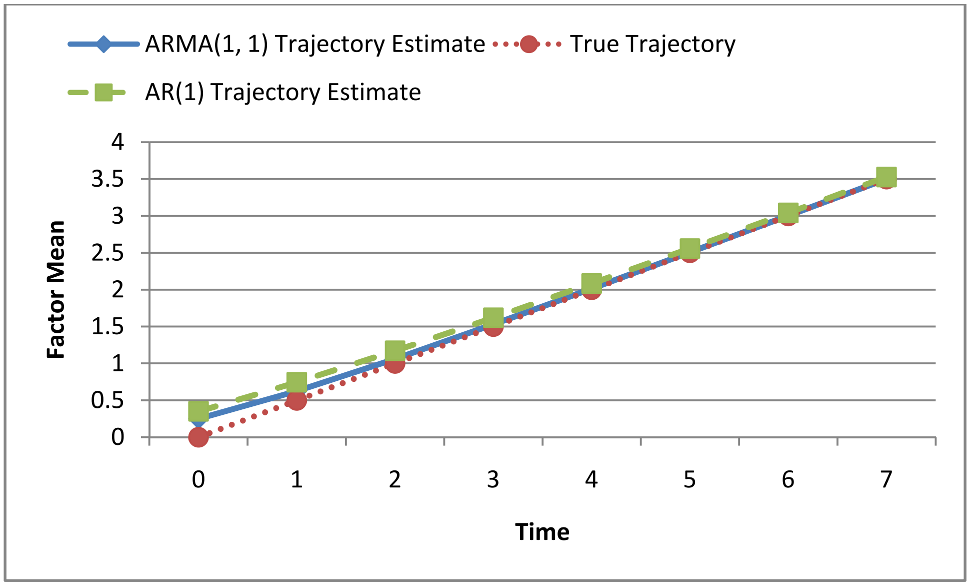

Figure 15. Comparison of AR(1) and ARMA(1, 1) estimated trajectories to the true trajectory.

Although the growth trajectory was estimated better than the growth parameter estimates indicated, there were other parameter estimates under the two models that were considered to be biased. In particular, the standard errors of the fixed effects were estimated poorly under conditions of small sample size and series length, and large magnitude autocorrelation (see Tables 20-21, 33-34). The variance components also were estimated poorly by the AR(1) and ARMA(1, 1) models across most non-zero serial correlation conditions. In general, large magnitude autocorrelation processes led to greater bias in the variance components estimates.

In addition to the magnitude of the autocorrelation process present in the data, the accuracy of the model estimate of the autocorrelation parameter appeared to influence the variance component estimates. When the autocorrelation was estimated accurately (e.g., when the sample size was 1000, the series length was 8, and the autocorrelation was

AR(1) where $\varphi = .3$), the variance of the level and the variance of the shape tended to be underestimated.

There were conditions under which the autocorrelation parameter was underestimated (e.g., when the series length was 8 and autocorrelation was ARMA(1, 1) where $\varphi = .8$ and $\theta = .3$). When the autocorrelation parameter was underestimated, the relative bias of the variance estimate decreased. Severe underestimation of the autocorrelation parameter resulted in overestimation of the variance parameters. An example of this trend can be seen clearly under the AR(1) condition where $\varphi$ was set to .3 and the series length was 5. As depicted in Figure 11, both the AR(1) and ARMA(1, 1) models overestimated the variance of the level when the sample size was 100; however, the discrepancy between the estimated and true parameter values decreased steadily as the sample size increased. When the sample size was 200, the AR(1) estimate was considered to be unbiased, and when the sample size was 500, the ARMA(1, 1) estimate was considered to be unbiased. When the sample size was 1,000, both variance estimates were underestimated. The autocorrelation parameter was underestimated most severely when the sample size was 100 and converged on the true parameter value as the sample size approached 1,000 under both models.

Although the direction of the bias in the variance components estimates under the AR(1) and ARMA(1, 1) models was usually the opposite of the direction of the bias seen under the curve-of-factors models (i.e., negative bias vs. positive bias), Hu and Bentler's (1999) suggested combination of the CFI, TLI, and RMSEA fit criteria indicated that the AR(1) and ARMA(1, 1) models fit the data well, also. When the sample size was at least 200, both models were indicated as having acceptable fit to the data either 100% of the time or very close to 100% of the time across each condition. Thus, as was seen with the curve-of-factors model, the models were evaluated as fitting the data acceptably despite parameter estimates that were considered to be biased.

A final aspect of the performance of the AR(1) and ARMA(1, 1) models that was worse than expected was the number of inadmissible solutions produced by each model. The inadmissible solutions occurred because the variance/covariance matrix of the level and shape was non-positive definite. A $p$ x $p$ matrix can be defined as non-positive definite if some of the matrix's $p$ eigenvalues are less than zero (Wothke, 1993). The number of inadmissible solutions produced in this study were higher under the AR(1) and ARMA(1, 1) models than under the curve-of-factors model across all conditions. There are several possible factors that can increase the probability that a matrix of variance/covariance estimates will be non-positive definite.

First, the probability of having a non-positive definite solution is higher if the sample size and/or the number of indicators is small (Boomsma, 1985). To test whether or not additional indicators would decrease the number of inadmissible solutions, I ran an additional 1,000 simulated data sets where each factor was indicated by 5 observed variables under the AR(1) condition where $\varphi = .8$ and the series length was 8. I chose to examine this particular condition because it produced the largest proportions of inadmissible solutions across sample sizes for both the AR(1) and ARMA(1, 1) models. Adding additional indicators did not decrease the number of inadmissible solutions.

Although the small number of indicators per factor did not appear to contribute to the number of inadmissible solutions, the sample size did appear to be a contributing factor, as the number of inadmissible solutions decreased as sample size increased across conditions. To test whether or not the inadmissible solutions would be eliminated provided a sufficient sample size, I reran 1,000 simulated data sets using sample sizes of 2,000, 5,000, and 10,000 again under the AR(1) condition where $\varphi = .8$ and the series length was 8. The results of this simulation are presented in Table 37.

Table 37.

*Proportion of Inadmissible Solutions for the ARMA(1, 1) and AR(1) Curve-of-factors Models while Varying Sample Size (N)*

| N | AR(1) | ARMA(1, 1) |
|---|---|---|
| 2000 | 0.200 | 0.105 |
| 5000 | 0.132 | 0.024 |
| 10000 | 0.146 | 0.001 |

As shown in Table 37, increasing the sample size led to a decrease in the number of inadmissible solutions for the ARMA(1, 1) model such that only one inadmissible solution was produced when the sample size was 10,000. The results for the AR(1) model were not as straightforward. The number of inadmissible solutions produced by the AR(1) model decreased as the sample size increased to 5,000; however, contrary to expectations, the number of inadmissible solutions was larger when the sample size was 10,000 than it was when the sample size was 5,000.

Other factors that can cause inadmissible solutions include outliers and/or non-normality in the data, too many parameters in the model, and empirical underidentification (Wothke, 1993). However, it seems unlikely that these factors were causing the AR(1) or ARMA(1, 1) model to produce inadmissible solutions.

All three of the examined models were analyzing the same data, so outliers should have affected the curve-of-factors model similarly to the AR(1) and ARMA(1, 1) models. Yet the curve-of-factors produced very few inadmissible solutions. The data were generated as multivariate normal, so non-normality should not have been an issue. If the number of parameters were the problem, then the ARMA(1, 1) model would be expected to have the largest number of inadmissible solutions, because it has the most parameters. For the same reason, empirical underidentification, which would occur if the randomly generated data set had a structure that was insufficient for the number of specified model parameters, should also affect the ARMA(1, 1) model more so than the AR(1) model. Closer inspection of the parameter estimates indicated that the variance of the first factor (i.e., the factor measured at the initial time point) was overestimated severely under the AR(1) and ARMA(1, 1) models in the presence of serial correlation. Future research into the causes of this overestimated variance parameter may be helpful.

Across all of the models investigated, the factor that most strongly influenced the relative bias of the estimates was the magnitude of the autocorrelation present in the data. In the absence of serial correlation, all of the methods tested produced unbiased parameter estimates of the fixed and variance components under the combination of large sample size and series length. When serial correlation was present in the data, all of the models produced biased variance components estimates. This poses potential problems for applied researchers in the social sciences, as it seems reasonable that growth and autocorrelation could be present in the same data set when subjects are measured repeatedly with the same instrument.

Because the reliability of an instrument can be defined as the extent to which its measurements remain consistent over repeated tests of the same subject under identical conditions, it is evident that the repeated measurement of a subject with a reliable instrument implies correlated measurements. Thus repeated measurements with a reliable instrument would seem to imply that the quasi-simplex model would be appropriate. On the other hand, the quasi-simplex model is a deterministic model that would not adequately describe varying trends in the data. Varying trends across time would be expected when collecting data in fields such as education, where the general consensus is that schools and teachers can affect students' growth trajectories. In short, it seems likely that in practice researchers would encounter longitudinal data sets containing both serial correlation and growth processes. This study has shown that the measurement of growth parameters of interest under such conditions can be problematic.

### *Limitations and Suggestions for Future Research*

The data simulation and analyses conducted in the study have provided some indication of the influence of serial correlation on the estimation of growth parameters under the curve-of-factors model. However, a simulation study by design offers a limited set of conditions, and therefore omits other conditions that could produce interesting or contradictory results. For example, an examination of the performance of the ARMA(1, 1) model while increasing the sample size substantially may be worthy of future investigation.

One limitation of this study is that it only examined conditions where the measurement model was fixed across conditions. In particular, each factor was indicated by four observed variables, the item parameters were parallel within a factor, and strict factorial invariance was observed across measurement occasions. Future research could examine the impact of serial correlation on growth while varying the number of manifest indicators, the item parameters (e.g., tau-equivalent or congeneric), and invariance conditions (e.g., weak or configural).

Another limitation of this study is that only linear growth was simulated. Including curvilinear growth would have extended the simulation beyond manageable proportions; however, an examination of the effects of serial correlation on non-linear functional forms of growth may be useful.

As mentioned earlier in the paper, the curve-of-factors framework does not adapt easily to autocorrelation specifications within the measurement model errors. Nevertheless, there are other methods for specifying serial correlation in SEM such as directly estimating the correlation among different errors. The effects of serial correlation in the measurement structure of first-order latent growth models has been studied (Sivo et al. 2005). The effects of serial correlation in the measurement structure of second-order latent growth models seems worthy of future research as well.

Finally, the AR(1) and ARMA(1, 1) curve-of-factors models examined in this study have not been described previously in the literature. It is possible that there are alternative specifications for these models that would enable better performance. More research into the causes of the biased estimates and inadmissible solutions seen in this study may be warranted.

## *General Conclusion*

Overall, the curve-of-factors model was the most adequate method of the three models tested for analyzing data with serial correlation and growth processes in the same data set. If researchers are interested only in the fixed effects of growth, then it appears that the curve-of-factors model is adequate and serial correlation can be ignored. However, if the variance components are of interest, serial correlation in the data would raise questions about inferences drawn on parameter estimates under the curve-of-factors model.

Efforts to measure growth and serial correlation within the same data set were minimally successful in that the models were able to reproduce an accurate estimate of the autocorrelation parameter and a reasonably accurate approximation of the trajectory. However, biased variance components, standard errors of the fixed effects, and large numbers of inadmissible cases prevent the AR(1) and ARMA(1, 1) curve-of-factors models from being useful to practitioners at this time.

## Appendix: SAS Code for Curve-of-factors Models

```
PROC CALIS DATA=DAT METHOD=MAX UCOV AUG OUTRAM=SEMOUT;
  LINEQS
      Y11=F1 + E11, Y21=Y21 INTERCEPT + L1 F1 + E21,
      Y31=Y21 INTERCEPT + L1 F1 + E31, Y41=Y21 INTERCEPT + L1 F1 + E41,
      Y12=F2 + E12, Y22=Y21 INTERCEPT + L1 F2 + E22,
      Y32=Y21 INTERCEPT + L1 F2 + E32, Y42=Y21 INTERCEPT + L1 F2 + E42,
      Y13=F3 + E13, Y23=Y21 INTERCEPT + L1 F3 + E23,
      Y33=Y21 INTERCEPT + L1 F3 + E33, Y43=Y21 INTERCEPT + L1 F3 + E43,
      Y14=F4 + E14, Y24=Y21 INTERCEPT + L1 F4 + E24,
      Y34=Y21 INTERCEPT + L1 F4 + E34, Y44=Y21 INTERCEPT + L1 F4 + E44,
      Y15=F5 + E15, Y25=Y21 INTERCEPT + L1 F5 + E25,
      Y35=Y21 INTERCEPT + L1 F5 + E35, Y45=Y21 INTERCEPT + L1 F5 + E45,
      Y16=F6 + E16, Y26=Y21 INTERCEPT + L1 F6 + E26,
      Y36=Y21 INTERCEPT + L1 F6 + E36, Y46=Y21 INTERCEPT + L1 F6 + E46,
      Y17=F7 + E17, Y27=Y21 INTERCEPT + L1 F7 + E27,
      Y37=Y21 INTERCEPT + L1 F7 + E37, Y47=Y21 INTERCEPT + L1 F7 + E47,
      Y18=F8 + E18, Y28=Y21 INTERCEPT + L1 F8 + E28,
      Y38=Y21 INTERCEPT + L1 F8 + E38, Y48=Y21 INTERCEPT + L1 F8 + E48,
        F1= FA + 0 FB + D1,
        F2= FA + 1 FB + D2,
        F3= FA + 2 FB + D3,
        F4= FA + 3 FB + D4,
        F5= FA + 4 FB + D5,
        F6= FA + 5 FB + D6,
        F7= FA + 6 FB + D7,
        F8= FA + 7 FB + D8,
            FA=UA INTERCEPT + DA,
            FB=UB INTERCEPT + DB;
  STD
      E11 E21 E31 E41 E12 E22 E32 E42 E13 E23 E33 E43 E14 E24 E34 E44
      E15 E25 E35 E45 E16 E26 E36 E46 E17 E27 E37 E47 E18 E28 E38 E48 =
      VE11 VE21 VE31 VE41 VE12 VE22 VE32 VE42 VE13 VE23 VE33 VE43
      VE14 VE24 VE34 VE44 VE15 VE25 VE35 VE45 VE16 VE26 VE36 VE46
      VE17 VE27 VE37 VE47 VE18 VE28 VE38 VE48,
      D1-D8=VD1-VD8, DA DB=VDA VDB;
  COV
      DA DB=COV_AB;
  VAR
      Y11 Y21 Y31 Y41 Y12 Y22 Y32 Y42 Y13 Y23 Y33 Y43 Y14 Y24 Y34 Y44
      Y15 Y25 Y35 Y45 Y16 Y26 Y36 Y46 Y17 Y27 Y37 Y47 Y18 Y28 Y38 Y48;
RUN;
```

Curve-of-factors model with AR(1) parameter:

```
PROC CALIS DATA=DAT METHOD=MAX UCOV AUG OUTRAM=SEMOUT;
  LINEQS
      Y11=F1 + E11, Y21=L1 F1 + E21, Y31=L1 F1 + E31, Y41=L1 F1 + E41,
      Y12=F2 + E12, Y22=Y22 INTERCEPT + L1 F2 + E22,
      Y32=Y22 INTERCEPT + L1 F2 + E32, Y42=Y22 INTERCEPT + L1 F2 + E42,
      Y13=F3 + E13, Y23=Y22 INTERCEPT + L1 F3 + E23,
      Y33=Y22 INTERCEPT + L1 F3 + E33, Y43=Y22 INTERCEPT + L1 F3 + E43,
      Y14=F4 + E14, Y24=Y22 INTERCEPT + L1 F4 + E24,
      Y34=Y22 INTERCEPT + L1 F4 + E34, Y44=Y22 INTERCEPT + L1 F4 + E44,
      Y15=F5 + E15, Y25=Y22 INTERCEPT + L1 F5 + E25,
      Y35=Y22 INTERCEPT + L1 F5 + E35, Y45=Y22 INTERCEPT + L1 F5 + E45,
      Y16=F6 + E16, Y26=Y22 INTERCEPT + L1 F6 + E26,
      Y36=Y22 INTERCEPT + L1 F6 + E36, Y46=Y22 INTERCEPT + L1 F6 + E46,
      Y17=F7 + E17, Y27=Y22 INTERCEPT + L1 F7 + E27,
      Y37=Y22 INTERCEPT + L1 F7 + E37, Y47=Y22 INTERCEPT + L1 F7 + E47,
      Y18=F8 + E18, Y28=Y22 INTERCEPT + L1 F8 + E28,
      Y38=Y22 INTERCEPT + L1 F8 + E38, Y48=Y22 INTERCEPT + L1 F8 + E48,
        F1= D1,
        F2=PHI1 F1 + FA + 1 FB + D2,
        F3=PHI1 F2 + FA + 2 FB + D3,
        F4=PHI1 F3 + FA + 3 FB + D4,
        F5=PHI1 F4 + FA + 4 FB + D5,
        F6=PHI1 F5 + FA + 5 FB + D6,
        F7=PHI1 F6 + FA + 6 FB + D7,
        F8=PHI1 F7 + FA + 7 FB + D8,
            FA=UA INTERCEPT + DA,
            FB=UB INTERCEPT + DB;
  STD
      E11 E21 E31 E41 E12 E22 E32 E42 E13 E23 E33 E43 E14 E24 E34 E44
      E15 E25 E35 E45 E16 E26 E36 E46 E17 E27 E37 E47 E18 E28 E38 E48 =
      VE11 VE21 VE31 VE41 VE12 VE22 VE32 VE42 VE13 VE23 VE33 VE43
      VE14 VE24 VE34 VE44 VE15 VE25 VE35 VE45 VE16 VE26 VE36 VE46
      VE17 VE27 VE37 VE47 VE18 VE28 VE38 VE48,
      D1-D8=VD1-VD8, DA DB=VDA VDB;
  COV
      DA DB=COV_AB, D1 DA=COV_1A, D1 DB=COV_1B;
  VAR
      Y11 Y21 Y31 Y41 Y12 Y22 Y32 Y42 Y13 Y23 Y33 Y43 Y14 Y24 Y34 Y44
      Y15 Y25 Y35 Y45 Y16 Y26 Y36 Y46 Y17 Y27 Y37 Y47 Y18 Y28 Y38 Y48;
  BOUNDS -1 <= PHI1 <= 1;
RUN;
```

Curve-of-factors model with ARMA(1, 1) parameters:

```
PROC CALIS DATA=DAT METHOD=MAX UCOV AUG OUTRAM=SEMOUT;
  LINEQS
      Y11=F1 + E11, Y21=L1 F1 + E21, Y31=L1 F1 + E31, Y41=L1 F1 + E41,
      Y12=F2 + E12, Y22=Y22 INTERCEPT + L1 F2 + E22,
      Y32=Y22 INTERCEPT + L1 F2 + E32, Y42=Y22 INTERCEPT + L1 F2 + E42,
      Y13=F3 + E13, Y23=Y22 INTERCEPT + L1 F3 + E23,
      Y33=Y22 INTERCEPT + L1 F3 + E33, Y43=Y22 INTERCEPT + L1 F3 + E43,
      Y14=F4 + E14, Y24=Y22 INTERCEPT + L1 F4 + E24,
      Y34=Y22 INTERCEPT + L1 F4 + E34, Y44=Y22 INTERCEPT + L1 F4 + E44,
      Y15=F5 + E15, Y25=Y22 INTERCEPT + L1 F5 + E25,
      Y35=Y22 INTERCEPT + L1 F5 + E35, Y45=Y22 INTERCEPT + L1 F5 + E45,
      Y16=F6 + E16, Y26=Y22 INTERCEPT + L1 F6 + E26,
      Y36=Y22 INTERCEPT + L1 F6 + E36, Y46=Y22 INTERCEPT + L1 F6 + E46,
      Y17=F7 + E17, Y27=Y22 INTERCEPT + L1 F7 + E27,
      Y37=Y22 INTERCEPT + L1 F7 + E37, Y47=Y22 INTERCEPT + L1 F7 + E47,
      Y18=F8 + E18, Y28=Y22 INTERCEPT + L1 F8 + E28,
      Y38=Y22 INTERCEPT + L1 F8 + E38, Y48=Y22 INTERCEPT + L1 F8 + E48,
        F1= F9,
        F2=PHI1 F1 + FA + 1 FB + F10 + THETA1 F9,
        F3=PHI1 F2 + FA + 2 FB + F11 + THETA1 F10,
        F4=PHI1 F3 + FA + 3 FB + F12 + THETA1 F11,
        F5=PHI1 F4 + FA + 4 FB + F13 + THETA1 F12,
        F6=PHI1 F5 + FA + 5 FB + F14 + THETA1 F13,
        F7=PHI1 F6 + FA + 6 FB + F15 + THETA1 F14,
        F8=PHI1 F7 + FA + 7 FB + THETA1 F15 + F16,
            FA=UA INTERCEPT + DA,
            FB=UB INTERCEPT + DB;
  STD
      E11 E21 E31 E41 E12 E22 E32 E42 E13 E23 E33 E43 E14 E24 E34 E44
      E15 E25 E35 E45 E16 E26 E36 E46 E17 E27 E37 E47 E18 E28 E38 E48 =
      VE11 VE21 VE31 VE41 VE12 VE22 VE32 VE42 VE13 VE23 VE33 VE43
      VE14 VE24 VE34 VE44 VE15 VE25 VE35 VE45 VE16 VE26 VE36 VE46
      VE17 VE27 VE37 VE47 VE18 VE28 VE38 VE48,
      DA DB=VDA VDB, F9-F16=VD1-VD8;
  COV
      DA DB=COV_AB, F9 DA=COV_1A, F9 DB=COV_1B;
  VAR
      Y11 Y21 Y31 Y41 Y12 Y22 Y32 Y42 Y13 Y23 Y33 Y43 Y14 Y24 Y34 Y44
      Y15 Y25 Y35 Y45 Y16 Y26 Y36 Y46 Y17 Y27 Y37 Y47 Y18 Y28 Y38 Y48;
  BOUNDS -1 <= PHI1 THETA1 <= 1;
RUN;
```

# References


Bast, J., & Reitsma, P. (1997). Matthew effects in reading: A comparison of latent growth curve models with structured means. *Multivariate Behavioral Research, 32*(2), 135-167.

Bentler, P. M. (1980). Multivariate analysis with latent variables: Causal modeling. *Annual Review of Psychology, 31*, 419-456.

Bentler, P. M. (1988). Causal modeling via structural equation systems. In J. R. Nesselroade & R. B. Cattell (Eds.), *Handbook of multivariate experimental psychology* (2nd ed., pp. 317-333). New York: Plenum Press.

Bollen, K. A. (1989). *Structural equations with latent variables*. Hoboken, NJ: John Wiley & Sons.

Bollen, K. A. (2004). *Latent curve models: A structural equation perspective*. Hoboken, NJ: John Wiley & Sons.

Bollen, K. A., & Curran, P. J. (2004). Autoregressive latent trajectory (ALT) models a synthesis of two traditions. *Sociological Methods & Research, 32*(3), 336-383.

Boomsma, A. (1985). Nonconvergence, improper solutions, and starting values in Lisrel maximum likelihood estimation. *Psychometrika, 50*(2), 229-242.

Box, G. E., & Jenkins, G. M. (1976). *Time-series analysis: Forecasting and control*. San Francisco: Holden-Day.

Browning, C. R., Leventhal, T., & Brooks-Gunn, J. (2004). Neighborhood context and racial differences in early adolescent sexual activity. *Demography, 41*(4), 697-720.

Bunting, B. P., & Adamson, G. (2000). Assessing reliability and validity in the context of planned incomplete data structures for multitrait-multimethod models . *Developments in Survey Methodology, 15*, 37-53.

Chatterji, M. (2006). Reading achievement gaps, correlates, and moderators of early reading achievement: Evidence from the Early Childhood Longitudinal Study (ECLS) kindergarten to first grade sample. *Journal of Educational Psychology, 98*(3), 489-507.

Curran, P. J. (2000). A latent curve framework for studying developmental trajectories of adolescent substance abuse. In J. Rose, L. Chassin, C. Presson, & J. Sherman (Eds.), *Multivariate applications in substance use research* (pp. 1-42). Hillsdale, NJ: Erlbaum.

Curran, P. J., & Bollen, K. A. (2001). The best of both worlds: Combining autoregressive and latent curve models. In L. M. Collins & A. G. Sayer (Eds.), *New methods for the analysis of change* (pp. 107-135). Washington, DC: American Psychological Association.

DeShon, R. P. (1998). A cautionary note on measurement error corrections in structural equation models. *Psychological Methods, 3*(4), 412-423.

Fan, X., Felsövályi, A., Sivo, S. A., & Keenan, S. C. (2001). *SAS for Monte Carlo studies: A guide for quantitative researchers*. Cary, NC: SAS Institute Inc.

Ferron, J., Dailey, R., & Yi, Q. (2002). Effects of misspecifying the first-level error structure in two-level models of change. *Multivariate Behavioral Research, 37*, 379-403. Retrieved November 30, 2006, from Lawrence Erlbaum Associates, Inc. database.

Garner, C. L., & Raudenbush, S. W. (1991). Neighborhood effects on educational attainment: A multilevel analysis. *Sociology of Education, 64*(4), 251-262.

Guttman, L. (1954). A new approach to factor analysis: The radix. In P. F. Lazarfeld (Ed.), *Mathematical thinking in the social sciences* (pp. 258 – 348). New York: Columbia University Press.

Hamaker, E., Dolan, C., & Molenaar, P. (2002). On the nature of SEM estimates of ARMA parameters. *Structural Equation Modeling, 9*, 347-368.

Hamilton, J., Gagné, P. E., & Hancock, G. R. (2003, April). *The effect of sample size on latent growth models*. Paper presented at the Annual Meeting of the American Educational Research Association, Chicago, IL.

Hancock, G. R., Kuo, W., & Lawrence, F. R. (2001). An illustration of second-order latent growth models. *Structural Equation Modeling, 8*(3), 470-489.

Hauk, W. W., & Anderson, S. (1984). A survey regarding the reporting of simulation studies. *American Statistician, 38*, 214-16.

Hertzog, C., & Nesselroade, J. R. (2003). Assessing psychological change in adulthood: An overview of methodological issues. *Psychology and Aging, 18*(4), 639-657.

Hoogland, J. J., & Boomsma, A. (1998). Robustness studies in covariance structure modeling. *Sociological Methods and Research, 26*(3), 329-367.

Hu, L.-T., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling, 6*(1), 1-55.

Hussong, A. M., Hicks, R. E., Levy, S. A., & Curran, P. J. (2001). Specifying relations between affect and heavy alcohol use among young adults. *Journal of Abnormal Psychology, 110*, 449-461.

Jöreskog, K. G. (1988). Analysis of covariance structures. In J. Nesselroade & R. B. Cattell (Eds.), Handbook of multivariate experimental psychology (pp. 207 – 230). New York: Plenum.

Kenny, D. A., & Campbell, D. T. (1989). On the measurement of stability in over-time data. *Journal of Personality, 57*, 445-481.

Krull, J. L., & MacKinnon, D. P. (1999). Multilevel mediation modeling in group-based intervention studies. *Evaluation Review,* 23(4), 418-444.

Kwok, O.-M., West, S. G., & Green, S. B. (2007). The impact of misspecifying the within-subject covariance structure in multiwave longitudinal multilevel models: A Monte Carlo study. *Multivariate Behavioral Research, 42*(3), 557-592.

Leite, W. L. (2007). A comparison of latent growth models for constructs measured by multiple items. *Structural Equation Modeling, 14*(4), 581-610.

MacCallum, R. C., Kim, C., Malarkey, W. B., & Kiecolt-Glaser, J. K. (1997). Studying multivariate change using multilevel models and latent curve models. *Multivariate Behavioral Research, 32*(3), 215-253.

Marsh, H. W. (1993). Stability of individual differences in multiwave panel studies: Comparison of simplex models and one-factor models. *Journal of Educational Measurement, 30*, 157-183.

Meredith, W. (1993). Measurement invariance, factor analysis, and factorial invariance. *Psychometrika, 58*, 525-543.

Murphy, D. L., & Pituch, K. A. (2009). The performance of multilevel growth curve models under an autoregressive moving average process. *The Journal of Experimental Education, 77*(3), 255-282.

Reuterberg, S.-E., & Gustafsson, J.-E. (1992). Confirmatory factor analysis and reliability: Testing measurement model assumptions. *Educational and Psychological Measurement, 52*, 795-811.

Rogosa, D., & Willett, J. B. (1985). Satisfying simplex structure is simpler than it should be. *Journal of Educational Statistics, 10*, 99-107.

Rovine, M. J., & Molenaar, P. C. M. (2005). Relating factor models for longitudinal data to quasi-simplex and NARMA models. *Multivariate Behavioral Research, 40*(1), 83-114.

Rudinger, G., & Reitz, C. (2001). Structural equation modeling in longitudinal research on aging. In J. E. Birren & K. W. Schaie (Eds.), *Handbook of the psychology of aging* (5th ed., pp. 29-52). San Diego, CA: Academic Press.

Sampson, R. J., Raudenbush, S., & Earls, F. (1997). Neighborhoods and violent crime: A multilevel study of collective efficacy. *Science, 277*, 918-924.

SAS (Version 9.1) [Computer software]. (2005). Cary, NC: SAS Institute.

Singer, J. D., & Willett, J. B. (2003). *Applied longitudinal data analysis: Modeling change and event occurrence*. New York, NY: Oxford University Press.

Sivo, S., Fan, X., & Witta, L. (2005). The biasing effects of unmodeled ARMA time series processes on latent growth curve model estimates. *Structural Equation Modeling, 12*(2), 215-231. Retrieved December 28, 2006, from Lawrence Erlbaum Associates, Inc. database.

Sivo, S. A. (2001). Multiple indicator stationary time series models. *Structural Equation Modeling, 8*(4), 599-612.

Sivo, S. A., & Willson, V. L. (2000). Modeling causal error structures in longitudinal panel data: A Monte Carlo study. *Structural Equation Modeling, 7*, 174-205.

Stoolmiller, M. (1995). Using latent growth curve models to study developmental processes. In J. M. Gottman (Ed.), *The analysis of change* (pp. 103-138). Mahwah, NJ: Erlbaum.

Tisak, J., & Meredith, W. (1990). Descriptive and associative developmental models. In A. von Eye (Ed.), *Statistical methods in longitudinal research* (Vol. 2, pp. 387-406). San Diego, CA: Academic Press.

Wackerly, D. D., Mendenhall, W., & Scheaffer, R. L. (1996). *Mathematical statistics with applications*. Belmont, CA: Duxbury Press.

Willett, J. B., & Sayer, A. G. (1994). Using covariance structure analysis to detect correlates and predictors of individual change over time. *Psychological Bulletin, 116*, 363-381.

Wothke, W. (1993). Nonpositive definite matrices in structural equation modeling. In K. A. Bollen & J. S. Long (Eds.), *Testing structural equation models* (pp. 256-293). Newbury Park, CA: Sage.

# Vita

Daniel Lee Murphy was born in Manhattan, Kansas on July 22, 1961, the son of Robert and Edna Murphy. After graduating from Richardson High School, Richardson, Texas, he entered The University of Texas at Austin, Texas. He received a Bachelor of Arts degree in Psychology from The University of Texas at Austin in December, 1984, and he received an elementary education teaching certificate from The University of Texas at Austin in August, 1989. He began teaching in Austin area schools in 1989. He received a Master of Education degree in Guidance and Counseling from Southwest Texas State University in San Marcos, Texas, in 1997. During the following years he was a guidance counselor in Austin area public schools. In June, 2005, he entered the graduate school at The University of Texas at Austin. His publications include:

Murphy, D. L. and Pituch, K. A. (2009). The Performance of Multilevel Growth Curve Models under an Autoregressive Moving Average Process. *Journal of Experimental Education, 77*(3), 255-282.

Pituch, K. A., Tate, R. L., and Murphy, D. L. (in press). Three-Level Models for Indirect Effects in School- and Class-Randomized Experiments in Education. *Journal of Experimental Education*.

Permanent address: 5105 Kings Hwy, Austin, Texas 78745

This dissertation was typed by the author.